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To

God & My Parents

Vogan diagrams of hyperbolic Kac-Moody algebras, Kac-Moody superalgebras and some studies on root system of Kac-Moody superalgebra

by

Biswajit Ransingh

Ph.D. Student in Mathematics
Roll No. 507MA001
National Institute of Technology
Rourkela - 769008 (India)

A thesis in conformity with the requirements for

the Degree of Doctor of Philosophy in

National Institute of Technology, Rourkela.

2013

CERTIFICATE

This is to certify that the thesis entitled "**Vogan diagrams of hyperbolic Kac-Moody algebras, Kac-Moody superalgebras and some studies on root system of Lie superalgebras**" which is being submitted by *Biswajit Ransingh*, a Ph.D. Student in Mathematics, Roll No. 507MA001, National Institute of Technology, Rourkela - 769008 (India), for the award of the Degree of Doctor of Philosophy in Mathematics from National Institute of Technology, Rourkela is a record of bonafide research work done by him under my supervision. The results embodied in the thesis are new and have not been submitted to any other University or Institution for the award of any Degree or Diploma.

Professor K. C. Pati
Supervisor
Department of Mathematics
National Institue of Technology
Rourkela (India) -769008

ACKNOWLEDGEMENTS

I wish to acknowledge my sincere gratitude to my supervisor Dr. Kishore Chandra Pati, Professor of Mathematics, National Insitute of Technology, Rourkela for his inspiration in the preparation of this thesis.

I also take this opportunity to acknowledge quite explicitly and with gratitude my debt to Dr. Akrur Behera, Professor of Mathematics, National Institute of Technology, Rourkela for reading the entire thesis and giving some valuable suggestions and for what I know about Algebra, Homotopy theory etc., that I learned from him in the last five years.

I give my heartful thanks National Board of Higher Mathematics, India (Project Grant No. 48/3/2008-R&D II/196-R with P.I. Prof. K. C. Pati) for financial support.

I thank Professor Dimitry Leites, Stockholm University, Sweden and Professor Vladimir Lykshovia, High Energy Physics, St Petersberg, Russia for reading some of my work of the thesis and their useful suggestions. I show my thankfullness to Professor Leonard Scott, University of Virginia, USA for his source of inspiration.

I take the opportunity to express my thankfullness and gratitute to Dr. K Raghavan and Dr. S Viswanath, Professor of Mathematics, IMSc, Chennai for their constant encourgament and Dr. Punita Batra , Professor in Harish Chandra Research Institute (HRI, Allahabad) for her hospitality during the stay in HRI and valuable discussion in Vogan diagrams.

I am greatly indebted to my parents for bearing the inconvinience during my thesis work.

Biswajit Ransingh
Ph.D. Student in Mathematics
Roll No. 507MA001
National Institute of Technology
Rourkela - 769008 (India)

ABSTRACT

A Vogan diagram is a Dynkin diagram of triplet $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\bar{0}}, \triangle^{+})$, where $\mathfrak{g}_{\mathbb{R}}$ is a real Lie algebras, $\mathfrak{h}_{\bar{0}}$ Cartan subalgebra, $\triangle^{+}$ positive root system. Vogan diagrams are useful tools to classify the real forms of a Lie algebras, affine Kac-Moody algebras (both twisted and untwised). In our thesis we have classified Vogan diagrams for some hyperbolic Kac-Moody algebras which have potential physical application. The real forms of Lie Superalgebra, affine twisted and untwisted Kac-Moody superalgebras are also classified by Vogan diagrams. In our last Chapter we have also included the construction of splints of Lie superalgebras and discussed defining relations and flip super dynkin diagrams related with root system.

Contents

Chapter 0

Introduction

Kac-Moody algebras and superalgebras have played an increasingly crucial role in various areas of mathematics as well as theoretical physics. The hyperbolic Kac-Moody algebras which constitute a subclass of Lorentzian Kac-Moody algebras and some of their (almost split) real forms have appeared in a variety of problems in string field theory and supergravity theories. The almost split real forms have been classified by Tits-Satake diagrams [4]. The Vogan diagram is another way of clasification of real form and it is named after David Vogan. For a complex semi-simple Lie algebra $\mathfrak{g}$, it is well known that the conjugacy classes of real forms of $\mathfrak{g}$ are in one to one correspondence with the conjugacy classes of involutions of $\mathfrak{g}$, if we associate a real form $\mathfrak{g}_{\mathbb{R}}$ to one of its Cartan involutions θ. Using a suitable pair $(\mathfrak{h}, \Pi)$ of a Cartan subalgebra $\mathfrak{h}$ and a basis Π of the associated root system, an involution is described by a *Vogan diagram*; the equivalences of Vogan diagrams describe how such a diagram changes when one changes $(\mathfrak{h}, \Pi)$.

For an affine Kac-Moody algebra $\mathfrak{g}$, the one to one correpondence $\mathfrak{g}_{\mathbb{R}} \leftrightarrow \theta$ was established by Ben Messaoud and Rousseau ([52], Theorem 3.1). Knapp [36] brought Vogan diagram into the light to represent the real forms of the complex simple Lie algebras. Batra [2, 3] developed a corresponding theory of Vogan diagram for real forms of nontwisted affine Kac-Moody algebras, where the author used the involution of the Kobashi's famous paper [38] on automorphism of finite order of the affine Lie algebra $A_l^{(1)}$. The sequel of the Batra's article gave a broad sense of classification of invariants of real forms of affine untwisted Kac-Moody algebras. Tanushree developed the theory of Vogan diagram for affine twisted Kac-Moody algebras [48] In Chapter 2 we have studied hyperbolic Kac-Moody algebras. Hyperbolic Kac-Moody algebras among all types of Kac-Moody algebras is least studied, although these types of algebras have led to many potential physical applications.

One of the prototype for this is in the space like singularity, at each spatial point, the degree of freedom that carries the essential dynamics are the logarithms $\beta^i \equiv -\ln a_i$ of the scale factors a_i along a set of (special) independent spatial directions $(ds^2 = -dt^2 + \sum_i a_i^2(t,x)(w^i)^2)$. The billiard analysis refers to the dynamics, in the vicinity of a spacelike singularity, of a gravitational model described by the Lagrangian is given by [18]

$$\begin{aligned} \mathcal{L}_D &= {}^{(D)}R\sqrt{|{}^{(D)}g|}\, dx^0 \wedge ... \wedge dx^{D-1} - \sum_\alpha \star d\phi^\alpha \wedge d\phi^\alpha - \\ &\quad -\frac{1}{2}\sum_p e^{\sum_\alpha \lambda_\alpha^{(p)}\phi^\alpha} \star F^{(p+1)} \wedge F^{(p+1)}, \quad D \geq 3 \end{aligned}$$

where R is the spatial curvature scalar and $F^{(p)}$ is the field strength.

The real parameter $\lambda_\alpha^{(p)}$ measures the strength of the coupling to the dilaton. The dilatons are denoted by $\phi^\alpha, (\alpha = 1, ..., N)$; their kinetic terms are normalized with a weight 1 with respect to the Ricci scalar. The Einstein metric has Lorentz signature $(-, +, ..., +)$; its determinant is ${}^{(D)}g$.

The light-like wall and the walls bounding the billiard have different origins: some arise from the Einstein-Hilbert action and involve only the scale factors β^i, $(i = 1, ..., d)$, introduced through the Iwasawa decomposition of the space metric. For example a criterion for gravitational dynamics to be chaotic is that the billiard has finite volume. This in turn stems from the remarkable property that the billiard can be identified with the fundamental Weyl chamber of an hyperbolic Kac-Moody algebra. In this thesis we have obtained the Vogan diagrams of some of the hyperbolic Kac-Moody algebras, so that these diagrams can lead to more interesting studies related to billiard, M-theory etc,. More particularly we have also obtained Vogan diagram of hyperbolic Kac-Moody algebra E_{10}. The E_{10} hyperbolic Kac-Moody algebras play a fundamental role in small tension expansion of M-theory and arithmetic chaos in superstring cosmology [16, 17].

A Vogan diagram is a Dynkin diagram with some additional information as follows. The 2-elements orbits under θ (Cartan Involution) are exhibited by joining the corresponding simple roots by a double arrow and the 1-element orbit is painted in black (respectively, not painted), if the corresponding imaginary simple root is noncompact (respectively compact). The Vogan diagrams provide us to handle the problem of classification of real form of Lie algebra [36] in a quicker way.

The classification of real semisimple Lie algebras includes maximally compact and split Cartan subalgebras. The Vogan diagram is based on the classification of maximally compact Cartan subalgebras. Recently similar work has been done using Vogan superdiagrams to classify the real forms of contragradient Lie superalgebras [13], where the extended Dynkin diagrams of Lie superalgebra is used. But our method uses the ordinary Dynkin diagram as done by Knapp [36]. In the Chapter 3 we shall reconstruct all the real forms of Lie superalgebras by Vogan diagrams which is a quicker one to classify than the former one.

The real form of Lie superalgebra is defined as a real Lie superalgebra such that its complexification is the original complex Lie superalgebra. It can be seen easily that every standard real form is naturally associated to an antilinear involutive automorphism of the complex Lie superalgebra [49]. Real forms of Lie superalgebras have a growing application in superstring theory, M-theory and other branches of theoretical physics. Magic triangle of M-theory by Satake diagram has been obainted in [50]. Similar supergravity theory can be obatined by Vogan diagrams. Keeping this view into account we have classified all the Vogan diagrams of affine twisted Kac-Moody superalgebras and some trivial Vogan diagrams of affine untwised Kac-Moody superalgebras. These may lead us a step towards diagramatic representation of affine Kac-Moody symmetric superspaces if they exists. The existence of affine Kac-Moody symmetric spaces have already proved by W. Freyn et. al [21] in the form of tame Frechet manifolds.

After finding the Vogan diagrams of simple Lie superalgebras and associating them with their real forms now we concentrate on for similar studies on affine Kac-Moody superalgebras. In Chapter 4 and 5 we have obtained the Vogan diagrams of real forms of affine untwised Kac-Moody superalgebras and twisted affine Lie superalgebra respectively.

Application of splint properties in simple Lie algebras drastically simplifies the calculation of branching coefficient. In [15] David A Ritcher defines the term splint and has classified the splints of root systems of Lie algebras. One of the ingredients of the term splints is the ideal of a root system embedding. The structure of a root systems is characterised by the additive properties of its corresponding system of positive roots. A splint of a root system of Δ is a partition $\Delta_1 \cup \Delta_2$ into two subsets, each of which have the additive, but not necessarily metrical, properties of root system. In our last Chapter i.e., Chapter 6 we develop the theory of splints of simple Lie superalgebras and also obtain definning relations between Dynkin diagrams and fip Dynkin diagrams.

Chapter 1

Preliminary

In this chapter we recall some definitions and known results on Kac-Moody algebras and superalgebras. Some more definitions and results are inlcuded in the relevant chapters. This chapter serves as the base and background for the study of subsequent chapters and we shall keep on referring back to it as and when required. The main objective is to establish notation, while more detailed information on these structures appears in the references.

1.1 Kac-Moody algebra

We recall the elementary results of Kac-Moody algebra.

Definition 1.1.1. (*Generalised Cartan Matrix* (*GCM*)) [32] An integral matrix $A = (a_{ij})_{i,j=1}^{r}$ is said to be *GCM* if

$$a_{ij} = \begin{cases} 2 & i = j \\ \leq 0 & i \neq j \end{cases}$$

and

$$a_{ij} = 0 \Rightarrow a_{ji} = 0$$

Definition 1.1.2. (*Kac-Moody Algebra*) [32] Let e_i, f_i and h_i denote the $3r$ Chevalley generators. The *Kac-Moody algebra* is defined as the algebra $\mathfrak{g}$ together with the following relations:

(a) $[h_i, h_j] = 0$,

(b) $[e_i, f_i] = h_i$,

(c) $[e_i, f_j] = 0$ if $i \neq j$,

(d) $[h_i, e_j] = a_{ij} e_j$,

(e) $[h_i, f_j] = -a_{ij} f_j$,

(f) $(\text{ad } e_i)^{1-a_{ij}} e_j = 0$ if $i \neq j$.

(g) $(\text{ad } f_i)^{1-a_{ij}} f_j = 0$ if $i \neq j$.

Definition 1.1.3. (*Dynkin Diagram*) [32] A Dynkin diagram associated with a $GCM\ A$ is a graph with the following properties:

(a) The Dynkin diagram has r vertices.

(b) When $a_{ij} a_{ji} = n \leq 4$, the vertices i and j are joined by n lines.

(c) if $a_{ij} a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, the vertices i and j are connected by $|a_{ij}|$ lines and these lines are equipped with an arrow pointing towards i if $|a_{ij}| > 1$.

(d) If $n > 4$, i and j are joined by a thick line on which we write $(|A_{ij}|, |A_{ji}|)$ with $|A_{ij}| \geq |A_{ji}|$. This $n > 4$ case will only concern us when we discuss rank-2 hyperbolic algebras.

Remark 1.1.4. It is clear that A is indcomposable if and only if Dynkin diagram is a connected graph. Then the relationship between a GCM, a *Dynkin diagram* and an *algebra* as in above relations is one to one.

Theorem 1.1.5. *Let A be a GCM , then one and only of the following three possibilites holds for both A and A^t:*

(ii) *(Finite type) det $A \neq 0$, there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$; The algebras formed by these Cartan matrices are called simple Lie algebras.*

(ii) *(Affine type) det $A = 0$; there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $v = 0$; The algebras formed by these Cartan matrices are called affine Kac-Moody algebras.*

(iii) *(Indefinite type) there exists $u > 0$ such that $Au < 0$; $Av \geq 0$, $v \geq 0$ imply $v = 0$. This is of wild case type and the subclass of these types are given by*

(a) *Lorentzian type : det$(A) < 0$ and A has exactly one negative egienvalue.*

(b) *Hyperbolic type : det$(A) < 0$ A is neither of finite nor affine type, Every proper, indecomposable principal submatrix is either of finite or affine type.*

The class of Lorentzian generalized Cartan matrices includes but is larger that the class of hyperbolic generalized Cartan matrices.

A Kac-Moody algebra associated to a GCM of affine type is called affine Kac-Moody algebra. The affine Kac-Moody algebra associated to a generalised Cartan marix of type $X_l^{(1)}$ (from Table 2 and 3) is called a *non-twisted affine Kac-Moody algebra*. Similarly affine Kac-Moody algebras associated to a generalised Cartan marix of type $X_l^{(n)}$ (here $n = 2$ or 3) (from Table 4) is called *twisted affine Lie algebras.*

Definition 1.1.6. A matrix A_{ij} is called indecomposable if after reordering of its indices, it cannot decomposed as a non-trivial direct sum.

Definition 1.1.7. An $n \times n$ matrix A is said to be symmetrizable if there exist an invertible diagonal matrix D and symmetric matrix S such that $A = DS$.

Definition 1.1.8. (*Hyperbolic Kac-Moody Algebra*) [32] A GCM A is called a *matrix of hyperbolic type* if it is indecomposable, symmetrizable of indefinite type and if every proper connected subdiagram of Dynkin diagram of A is of finite or affine type. The Kac-Moody algebra is called *hyperbolic Kac-Moody Algebra* if the GCM is of hyperbolic type.

The general strategy in searching for hyperbolic Dynkin diagrams of rank $r+1$ is as follows:

(i) Draw all possible Lie and/or affine (including semisimple) diagrams of rank r.

(ii) Add an extra root, trying all possible lengths

(iii) Try connecting the new root to the old ones in all ways consistent with a symmetrizable GCM.

(iv) Test the resulting diagram by removing any point to see whether it reduces to (perhaps a disconnected combinations of) known finite or affine algebras, the twisted ones being included among the latter. A diagram that survives the test is of the hyperbolic type.

1.2 Root systems and Dynkin diagrams of simple Lie algebras

Definition 1.2.1 (Roots). The nonzero generalized weights of ad$\mathfrak{h}$ on $\mathfrak{g}$ are called the *roots* of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The set of roots are denoted by $\triangle$.

The Lie algebra $\mathfrak{g}$ has a root space decomposition with respect to $\mathfrak{h}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \triangle} \mathfrak{g}_\alpha$$

The members of $\mathfrak{g}_\alpha$ are called root vectors for the root α. An *abstract root system* in a finite dimensional real inner product space V with inner product $<\dot{,}>$ and norm squared $|\dot{|}^2$ is a finite set $\triangle$ of nonzero elements of V such that

(a) $\triangle$ spans V,

(b) the orthogonal transformations $s_\alpha(\phi) = \phi - \frac{2<\phi,\alpha>}{|\alpha|^2}\alpha$, for $\alpha \in \triangle$, carry $\triangle$ to itself (for $\phi \in V$),

(c) $\frac{2<\beta,\alpha>}{|\alpha|^2}$ is an integer whenever α and β are in $\triangle$.

An abstract root system is said to be reduced if $\alpha \in \triangle$ implies $2\alpha \notin \triangle$

Definition 1.2.2 (*Simple root*). A root $\alpha > 0$ is called simple if it cannot be written as a sum $\alpha = \beta + \gamma$ where β, γ are positive roots.

(i) The root systems of A_n:

$$\triangle = \{\pm(e_i - e_j) : 0 \leq i < j \leq n\}.$$

The roots $\alpha_i = e_i - e_{i+1} (1 \leq i \leq n)$ form a basis Π of $\triangle$, where e_is are orthonormal basis.

(ii) The root systems of B_n:

$$\triangle = \{\pm(e_i \pm e_j) : 1 \leq i < j \leq n\} \cup \{\pm e_i : 1 \leq i \leq n\}.$$

The roots $\alpha_i = e_i - e_{i+1}(1 \leq i \leq l-1)$, $\alpha_l = e_l$ form a basis Π. where $\{e_i : i \in I\}$ is assumed to be orthonormal for any set I.

(iii) The root systems of C_n:

$$\triangle = \{\pm(e_i \pm e_j) : 1 \leq i < j \leq n\} \cup \{\pm 2e_i : 1 \leq i \leq n\}.$$

The roots $\alpha_i = e_i - e_{i+1}(1 \leq i \leq n-1)$, $\alpha_l = 2e_l$ form a basis Π of $\triangle$.

(iv) The root systems of D_n:

$$\triangle = \{\pm(e_i \pm e_j) : 1 \leq i < j \leq n\},$$

The roots $\alpha_i = e_i - e_{i+1}(1 \leq i \leq n-1)$, $\alpha_l = e_{l-1} - e_l$ form a basis Π of $\triangle$.

(ix) The root systems of E_6:

The simple root system is given by $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} = \{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1), e_2 + e_1, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4\}$. The root system

$$\triangle = \left\{ \pm(e_i \pm e_j)(1 \leq i < j \leq 5), \pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{v(i)} e_i) \ (\sum_{i=1}^{5} v(i) e_i) \text{ even} \right\}.$$

(viii) The root systems of E_7:

The simple root system $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\} = \{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1), e_2 + e_1, e_2 - e_1, e_3 - e_2, e_4 - e_3, e_5 - e_4, e_6 - e_5\}$. The root system $\triangle = \left\{ \pm(e_i \pm e_j)(1 \leq i < j \leq 6), \pm(e_7 - e_8), \pm\frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6}(-1)^{v(i)} e_i) \quad (\sum_{i=1}^{6} v(i) \text{ odd}) \right\}$.

(vii) The root systems of E_8:

$$\triangle = \left\{ \pm(e_i \pm e_j)(i < j), \quad \frac{1}{2}(e_8 + \sum_{i=1}^{7}(-1)^{v(i)}e_i) \quad (\sum_{i=1}^{7} v(i) \text{ even}) \right\}.$$

The simple roots are $\alpha_1 = \frac{1}{2}(e_1+e_8) - \frac{1}{2}(e_2+e_3+e_4+e_5+e_6+e_7)$, $\alpha_2 = e_1+e_2$, $\alpha_3 = e_2 - e_1$, $\alpha_4 = e_3 - e_2$, $\alpha_5 = e_4 - e_3$, $\alpha_6 = e_5 - e_4$, $\alpha_7 = e_6 - e_5$, $\alpha_8 = e_7 - e_6$.

(v) The root systems of F_4:

$$\triangle = \left\{ \pm e_i, \pm e_i \pm e_j (i < j), \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

For a basis of $\triangle$ we can take roots $\alpha_1 = e_2 - e_3$, $\alpha_2 = e_3 - e_4$, $\alpha_2 = e_4$, $\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$.

(vi) The root systems of $G(2)$:

$$\triangle = \{\pm(e_1-e_2), \pm(e_2-e_3), \pm(e_1-e_3), \pm(2e_1-e_2-e_3), \pm(e_1+e_2-2e_3), \pm(e_1-2e_2+e_3)\}.$$

The simple roots $\Pi = \alpha_1 = e_1 - e_2$, $\alpha_1 = -2e_1 + e_2 + e_3$.

<table>
<tr><th>Lie algebra</th><th>Dynkin diagram</th></tr>
<tr><td>A_n</td><td>
</td></tr>
<tr><td>B_n</td><td>
</td></tr>
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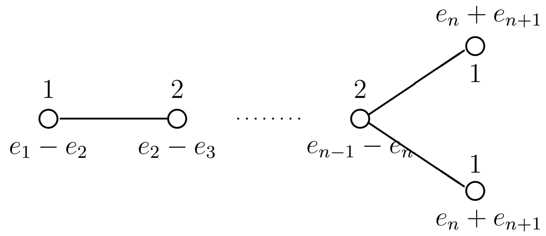
<tr><td>D_n</td><td>
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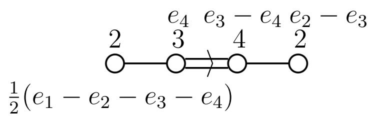
<tr><td>F_4</td><td>
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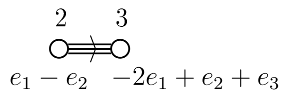
<tr><td>G_2</td><td>
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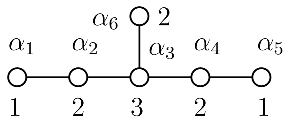
<tr><td>E_6</td><td>
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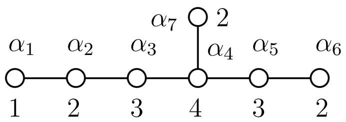
<tr><td>E_7</td><td>
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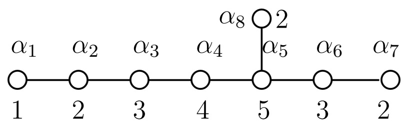
<tr><td>E_8</td><td>
</td></tr>
</table>

Table - 1

1.3 Root systems and Dynkin diagrams of affine Kac-Moody algebras

We can construct an infinite dimensional Lie algeba from a complex semisimple Lie algebra $\mathfrak{g}$ by considering the algebra $\mathbb{C}[t, t^{-1}]$ whose element are Laurent polynomials. So the algebra formed by this can be of the form

$$L(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$$

which is the Lie algebra with commutation relations

$$[P \otimes X, Q \otimes Y]_0 = PQ \otimes [X, Y]_0 : P, Q \in \mathbb{C}[t, t^{-1}], X, Y \in \mathbb{C}.$$

If K is the central element; then we can extend the loop algebra as

$$\hat{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbb{C}K.$$

In order to construct the affine Lie algebra $\hat{\mathfrak{g}}$ corresponding to the simple Lie algebra $\mathfrak{g}$, we have to extend the algebra $\hat{L}(\mathfrak{g})$. For this we add to $\hat{L}(\mathfrak{g})$, the element $D = t\frac{d}{dt}$ and define the commutation relation of D with the elements of $\hat{L}(\mathfrak{g})$ and with K by the formulas

$$[D, P \otimes X] = -[P \otimes X, D] = t\frac{dP}{dt} \otimes X,$$

$$[D, K] = -[K, D] = 0$$

Thus the affine Lie algebra $\hat{\mathfrak{g}}$ is the complex linear space $\hat{\mathfrak{g}}$

$$\hat{\mathfrak{g}} = L(\mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

If $\mathfrak{h}$ is the Cartan subalgebra of the complex simple Lie algebra $\mathfrak{g}$, then we define the subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$ as

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

We can now define the dual space $\mathfrak{h}_R^*$ of the linear forms for $\mathfrak{h}$. Let us spread forms λ from $\mathfrak{h}_R^*$ (and from all the dual spaces $\mathfrak{h}_R^*$ for $\mathfrak{h}$) onto the whole of the space $\hat{\approx}_R$ by putting $\lambda(K) = \lambda(\alpha) = 0$. We also introduce linear forms δ on $\hat{\mathfrak{h}}$ by letting $\delta(D) = 1$, $\delta(\mathfrak{h}) = 0$; $h \in \mathfrak{h}$, $\delta(K) = 0$, Then we obtain the space

$$\hat{\mathfrak{h}}_R^* = \mathfrak{h}_R^* \oplus \mathbb{C}\delta$$

of linear forms on $\hat{\mathfrak{h}}$. The root space decomposition of affine Lie algebra is of the form

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \sum_{\gamma \in \triangle} \hat{\mathfrak{g}}_\gamma,$$

where the set of root

$$\hat{\triangle} = \{m\delta | m \in Z/\{0\} \cup \{m\delta + \alpha | m \in Z\ , \alpha \in \triangle\}\},$$

where $\triangle$ is the set of roots for the Lie algebra $\mathfrak{g}$ and $\hat{\mathfrak{g}}_{m\delta} = t^{m\delta} \otimes h$, $\hat{\mathfrak{g}}_{\alpha+m\delta} = t^{m\delta} \otimes \mathfrak{g}_\alpha$. The roots $m\delta+\alpha$, $\alpha \in \triangle$ are called real and the roots $m\delta$ are termed as imaginary.

A minimal system of roots Π can be selected in $\hat{\triangle}$ such that every root $\gamma \in \hat{\triangle}$ is represented as a linear combination of the roots from Π. The set of simple roots of affine Lie algebra are $\{\alpha_0 = \delta - \theta, \alpha_1, \alpha_2, \cdots, \alpha_r\}$.

We will first shown the root system of affine untwisted Kac-Moody algebras, followed by its Dynkin diagrams.

(i) $A_l^{(1)}(l \geq 1)$: The root system is given as

$$\tilde{\triangle} = \{e_i - e_j + n\delta, m\delta, 1 \leq i, j \leq l+1, i \neq j, 0 \neq m \in Z, n \in Z\}$$

(ii) $B_l^{(1)}(l \geq 2)$: The root system is given as

$$\tilde{\triangle} = \left\{ \begin{array}{c} \tilde{\triangle} = \pm e_i \pm n\delta, \pm(e_i \pm e_j) + n\delta, m\delta, \\ 1 \leq i, j \leq l, i \leq j, 0 \neq m \in Z, n \in Z \end{array} \right\}$$

(iii) $C_l^{(1)}(l \geq 2)$: The root system is given as

$$\tilde{\triangle} = \left\{ \begin{array}{c} \tilde{\triangle} = \pm 2e_i \pm n\delta, \pm(e_i \pm e_j) + n\delta, m\delta, \\ 1 \leq i, j \neq l, 0 \neq m \in Z, n \in Z \end{array} \right\}$$

(iv) $D_l^{(1)}(l \geq 4)$: In this case the root system is

$$\tilde{\triangle} = \left\{ \begin{array}{c} \tilde{\triangle} = \pm(e_i \pm e_j) + n\delta, m\delta, \\ 1 \leq i, j \leq l, i \leq j, 0 \neq m \in Z, n \in Z \end{array} \right\}$$

(v) $E_6^{(1)}$: The root system is given as

$$\tilde{\triangle} = \left\{ \begin{array}{c} \pm(e_i \pm e_j) + n\delta(i \leq 1, j \leq 5), \\ \pm\frac{1}{2}(e_8 - e_7 - e_6 \sum\limits_{i=1}^{5}(-1)^{v(i)} e_i) + n\delta(\sum\limits_{i=1}^{5} v(i) e_i \text{ even }) \; i < j, n \in Z \end{array} \right\}$$

(vi) $E_7^{(1)}$: The root system is given as

$$\tilde{\triangle} = \left\{ \begin{array}{c} \pm(e_i \pm e_j) + n\delta(i \leq 1, j \leq 6), \; \pm(e_7 - e_8) + n\delta, \\ \pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum\limits_{i=1}^{6}(-1)^{v(i)} e_i) + n\delta(\sum\limits_{i=1}^{6} v(i) e_i \text{ even }) \; n \in Z \end{array} \right\}$$

(vii) $E_8^{(1)}$: The root system is given as

$$\tilde{\triangle} = \left\{ \begin{array}{c} \pm(e_i \pm e_j) + n\delta(i \leq j), \\ \sum\limits_{i=1}^{8}(-1)^{v(i)} e_i) + n\delta(\sum\limits_{i=1}^{8} v(i) e_i \text{ even }) \; i < j, n \in Z \end{array} \right\}$$

(viii) $F_4^{(1)}$: The root system is given as

$$\tilde{\triangle} = \left\{ \begin{array}{c} \pm(e_i \pm e_j) + n\delta \ , \pm(e_i \pm e_i + n\delta, \\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) + n\delta, (i < j), n \in Z \end{array} \right\}$$

(ix) $G_2^{(1)}$:

Let's put $\phi_i = e_i - \frac{1}{3}(e_1 + e_2 + e_3)$ $(i = 1, 2, 3)$. So that $\sum \phi_i = 0$. The root system of $G_2^{(1)}$ can be considered as

$$\tilde{\triangle} = \left\{ \begin{array}{c} \pm(\phi_i + n\delta \ (1 \leq i \leq 3), \ \pm(\phi_i - \phi_j) + n\delta, m\delta, \\ (1 \leq i \leq j \leq 3), n \in Z \end{array} \right\}$$

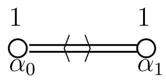
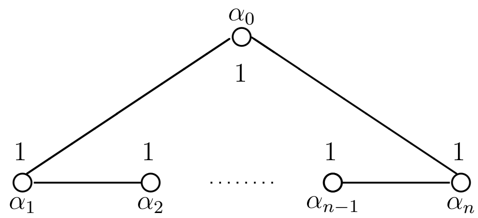
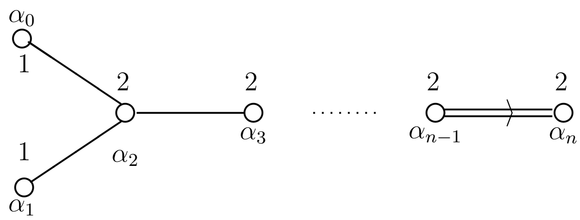
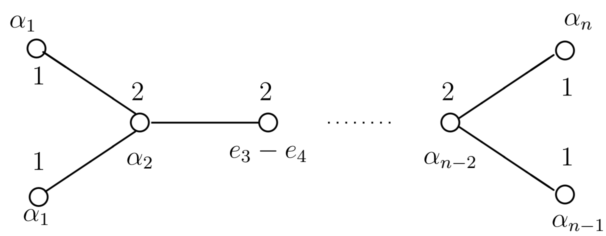
$\mathfrak{g}$	Dynkin diagram
$A_1^{(1)}$	
$A_n^{(1)}(n \geq 2)$	
$B_n^{(1)}(n \geq 3)$	
$C_n^{(1)}(n \geq 2)$	
$D_n^{(1)}(n \geq 4)$	

Table - 2

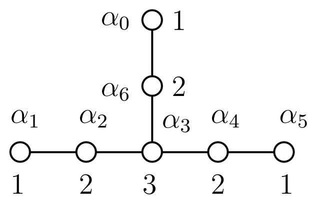
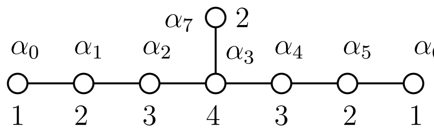
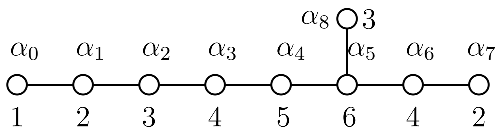
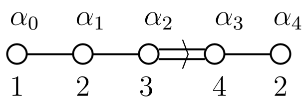
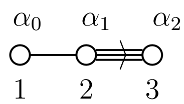
<table>
<tr><th>$\mathfrak{g}$</th><th>Dynkin diagram</th></tr>
<tr><td>E_6^1</td><td>
</td></tr>
<tr><td>E_7^1</td><td>
</td></tr>
<tr><td>E_8^1</td><td>
</td></tr>
<tr><td>$F_4^{(1)}$</td><td>
</td></tr>
<tr><td>$G_2^{(1)}$</td><td>
</td></tr>
</table>

Table - 3

Let $a_0, a_1, \cdots, a_l$ be the numerical labels of Dynkin diagrams in affine Kac-Moody algebras (Table 2, 3 and 4). These labels are the coordinates of the unique vector $\delta = (a_0, a_1, \cdots, a_l)$ such that $A\delta = 0$ and the a_i are positive relativiely prime integers. Then $a_0 = 1$ unless A is of type $A_{2l}^{(2)}$, in which case $a_0 = 2$

$$\delta = \sum_{i=0}^{l} a_i \alpha_i \in Q$$

Let A be a symmetrizable generalized Cartan matrix, and $(.|.)$ a standard invariant bilinear form. Then for a real root α we have $|\alpha|^2 = |\alpha_i|^2$ for some simple root α_i.

Remark 1.3.1. (a) If A is symmetric, then all simple roots and hence all real roots have the same square length.

(b) If A is not symmetric matrix from the Table of affine Kac-Moody algebras and A is not of type $A_{2l}^{(2)}$, then every real root is either short or long; for the type $A_{2l}^{(2)}$ there are real roots of three different lengths.

Proposition 1.3.2. *(a)* $\triangle^{re} = \{\alpha + n\delta | \alpha \in \triangle_s, n \in \mathbb{Z}\}$ *if* $k = 1$

(b) $\triangle^{re} = \{\alpha + n\delta | \alpha \in \triangle_s, n \in \mathbb{Z}\} \cup \{\alpha + n\delta | \alpha \in \triangle_l, n \in \mathbb{Z}\}$ *if* $k = 2$ *or* 3, *but* A *is not of type* A_{2l}^{2}.

(c) $\triangle^{re} = \left\{\frac{1}{2}(\alpha + (2n-1)\delta | \alpha \in \overset{0}{\triangle}_s, n \in \mathbb{Z}\right\} \cup \{\alpha + n\delta | \alpha \in \triangle_s, n \in \mathbb{Z}\}$ *if* $k = 2$ *or* $k = 3$, *but* A *is of type* A_{2l}^{2}.

(d) $\triangle^{re} + k\delta \subset \triangle^{re}$

(e) $\triangle_+^{re} = \{\alpha \in \triangle^{re}$ *with* $n > 0\} \cup \overset{0}{\triangle}_+$.

1.4 Dynkin diagrams of twisted affine Kac-Moody algebras

The root system of all the twisted affine Kac-Moody algebras are listed below in terms of the basis vectors e_i

(i) $A_2^{(2)}$: The root system for this case is

$$\tilde{\triangle} = \left\{ \begin{array}{c} \pm e_i \pm n\delta, \pm(e_i \pm e_j) + n\delta,\ (2n+1)\delta \pm 2e_i, m\delta, \\ 1 \leq i, , j \leq l, i < j, n \in Z^*, 0 \neq m \in Z \end{array} \right\}$$

(ii) $A_{2l-1}^{(2)}$: The root system for this case is given in terms of the rescaled folded roots

$$\tilde{\triangle} = \left\{ \begin{array}{c} \pm\frac{2\eta_i}{\sqrt{2}} + m\delta, \sqrt{\frac{1}{2}}(\pm\eta_i \pm \eta_j) + \frac{1}{2}m\delta, \frac{1}{2}n\delta, \\ 1 \leq i \neq j \leq nj, n \in Z^*, m \in Z \end{array} \right\}$$

where $\eta_i = \sqrt{(e_i - e_{2l+1-i})}\ \ (1 \leq i \leq l)$

(iii) $D_l^{(2)}$: The root system for this case is

$$\tilde{\triangle} = \left\{ \begin{array}{c} \pm e_i + \frac{1}{2}m\delta, \pm(e_i \pm e_j) + m\delta,\ \frac{1}{2}n\delta, \\ 2 \leq i \leq j, i < j, n \in Z, m \in Z \end{array} \right\}$$

(iv) $E_6^{(2)}$: The root system for this case is

$$\tilde{\triangle} = \left\{ \begin{array}{c} \pm\eta_i \pm \eta_j + m\delta, \pm\eta_i + m\frac{\delta}{2}, \frac{1}{2}(\pm\eta_1 \pm \eta_2 \pm \eta_3 \pm \eta_4) + \frac{1}{2}m\delta, \\ \frac{1}{2}n\delta, 1 \leq i \neq j \leq \triangle, m \in Z, n \in Z^* \end{array} \right\}$$

where $\eta_1 = \frac{1}{2}(e_5 - e_6 - (e_7 + e_8)$, $\eta_2 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$, $\eta_3 = \frac{1}{2}(-e_1 - e_2 + e_3 + e_4)$, $\eta_4 = \frac{1}{2}(-e_1 + e_2 - e_3 + e_4)$.

(v) $D_4^{(3)}$: The root system for this case is

$$\tilde{\triangle} = \left\{ \begin{array}{c} \pm(\eta_i - \eta_j) + m\delta, \pm\frac{1}{3}(2\eta_i - \eta_j - eta_k) + \frac{1}{3}m\delta, \\ \frac{1}{3}n\delta, 1 \leq i \neq j \neq k \leq 3, m \in Z, n \in Z^* \end{array} \right\}$$

where $\eta_1 = -e_4$, $\eta_4 = e_1$, $\eta_i = -e_i$, $(i \neq 1, 4)$.

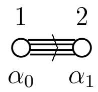
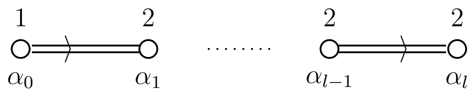
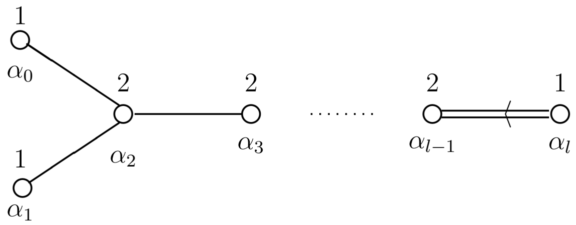
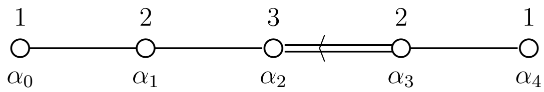
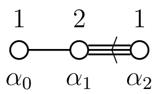
$\mathfrak{g}$	Dynkin diagram
$A_2^{(2)}(l \geq 2)$	
$A_{2l}^{(2)}(l \geq 2)$	
$A_{2l-1}^{(2)}(l \geq 3)$	
$D_{l+1}^{(2)}(l \geq 2)$	
$E_6^{(2)}(l \geq 2)$	
$G_2^{(3)}$	

Table - 4

The detail Dynkin diagram of hyperbolic Kac-Moody algebras are given in [54, 55].

1.5 Kac-Moody superalgebras

Now we briefly introduce the super symmetric version of Kac-Moody algebras called Kac-Moody superalgebras. Let τ be a subset $\{1, \cdots, n\}$. To a given generalized Cartan matrix A and subset τ, we associate a complex contragradient superalgebra $\mathcal{G}(A, \tau)$ called Kac-Moody superalgebra with $3n$ generators h_i, e_i, f_i and $\mathbb{Z}_2$ gradation defined by deg e_i = deg $f_i = \bar{0}$ if $i \neq \tau$, dege_i = deg$f_i = \bar{1}$ if

$i \in \tau$ and deg$h_i = \bar{0}$ for all i. The generators h_i, e_i and f_i are subject the following set of relations:

(a) $[h_i, h_j] = 0$,

(b) $[e_i, f_j] = \delta_{ij} h_i$

(c) $[h_i, e_j] = a_{ij} e_j$,

(d) $[h_i, f_j] = -a_{ij} f_j$,

(e) $(\text{ad } e_i)^{1-\tilde{a}_{ij}} e_j = 0$ if $i \neq j$.

(f) $(\text{ad } f_i)^{1-\tilde{a}_{ij}} f_j = 0$ if $i \neq j$.

where the matrix $\tilde{A} = (\tilde{a}_{ij})$ is deduced from the Cartan matrix $A = a_{ij}$ of $\mathcal{G}(A, \tau)$ by replacing all its positive off diagonal entries by -1.

The Kac-Moody superalgebra $\mathcal{G}(A, \tau)$ is associated with a Dynkin diagram according to the following rules. We assume that $i \in \tau$ if $a_{ii} = 0$.

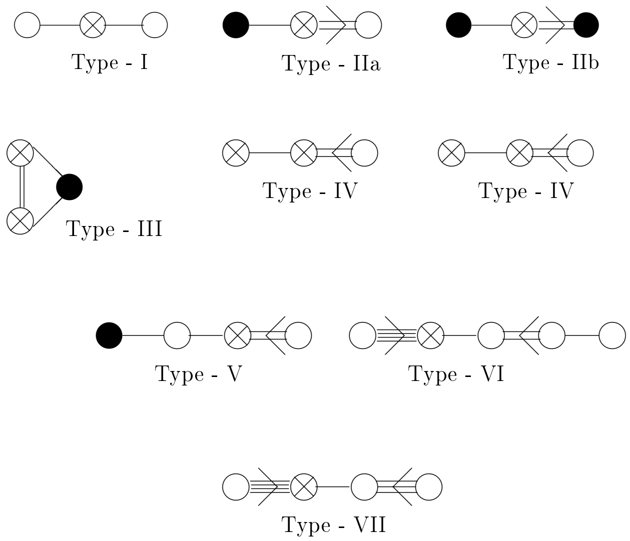
From a GCM, A with each i of the diagonal entries (a_{ii}) 2 and $i \notin \tau$ a white dot $\circ$ and $i \in \tau$ a black dot $\bullet$, to each i such that $a_{ii} = 0$ and $i \in \tau$ a grey dot $\otimes$. The i-th and j-th roots will be joined by $\zeta_{ij} = \max\Big(|a_{ij}|, |a_{ji}|\Big)$ lines with $|a_{ij}\, a_{ji}| \leq 4$ and the off digonal entries nonzero where for off diagonal entries zero; then the number of connection lines are $|a_{ij}| = |a_{ji}|$ with $|a_{ij}|$ and $|a_{ji}| \leq 4$

The arrows will be added on the lines connecting the i-th and j-th dots when $\zeta_{ij} > 1$ and $|a_{ij}| \neq |a_{ji}|$, pointing from j to i if $|a_{ij}| > 1$.

In the case of Lie algebras the matrices (a_{ij}) and $(\tilde{a}_{ij})$ coincide and the above gradation relations reduce to the standard Serre relations. However, in the case of superlagebra, the description given by the above Serre's relation leads in general to a bigger superalgebra than the superalgebra under consideration. So it is necessary to write supplementary relations involving more than two generators, in order to quotient the bigger superalgebra and recover the original one. These supplementary conditions appear when one deals with odd roots of zero length ($a_{ii} = 0$). The supplementary conditions depend on the different kind of vertices, which appear in the Dynkin diagarms. For example, in case of $A(m,n)$ if α_i is an odd root then the supplementary condition

$$[[[e_{i-1}, e_i], e_{i+1}], e_i] = 0$$

is necessary. For the superalgeba $A(m,n)$ with $m, n \geq 1$ and $B(m,n)$, $C(n+1)$, $D(m,n)$, the vertices can be of the following type:



where the black dot represents either white dots associated to even roots or grey dots associated to isotropic odd roots. The supplementary conditions take the following form:

(i) For the type I, IIa and IIb vertices:

$(\text{ ad } e_{\pm\alpha_m})(\text{ ad } e_{\pm\alpha_{m+1}})(\text{ ad } e_{\pm\alpha_m})e_{\pm\alpha_{m-1}}$

$= (\text{ ad } e_{\pm\alpha_m})(\text{ ad } e_{\pm\alpha_{m-1}})(\text{ ad } e_{\pm\alpha_m})e_{\pm\alpha_{m+1}} = 0$

(ii) $(\text{ ad } e_{\pm\alpha_m})(\text{ ad } e_{\pm\alpha_{m+1}})e_{\pm\alpha_{m-1}} = (\text{ ad } e_{\pm\alpha_m})(\text{ ad } e_{\pm\alpha_m})e_{\pm\alpha_{m+1}} = 0$

(iii) For the type IV vertex :

$(\text{ ad } e_{\pm\alpha_m})[(\text{ ad } e_{\pm\alpha_{m+1}})(\text{ ad } e_{\pm\alpha_m})e_{\pm\alpha_{m-1}}$

$(\text{ ad } e_{\pm\alpha_{m+1}})(\text{ ad } e_{\pm\alpha_{m-1}})] = 0$

(iv) For the type V vertex :

$(\text{ ad } e_{\pm\alpha_m})(\text{ ad } e_{\pm\alpha_{m-1}})(\text{ ad } e_{\pm\alpha_m})e_{\pm\alpha_{m+1}}$

$(\text{ ad } e_{\pm\alpha_m})(\text{ ad } e_{\pm\alpha_{m-1}}) \text{ ad } e_{\pm\alpha_{m-2}} = 0$

(v) For the type VI vertex :

$(\text{ ad } e_{\pm\alpha_2})(\text{ ad } e_{\pm\alpha_1})(\text{ ad } e_{\pm\alpha_3})e_{\pm\alpha_2}$

$(\text{ ad } e_{\pm\alpha_1}) \text{ ad } e_{\pm\alpha_0} - 2(\text{ ad } (e_{\pm\alpha_1})(\text{ ad } e_{\pm\alpha_2})(\text{ ad } e_{\pm\alpha_3})(\text{ ad } e_{\pm\alpha_1}) \text{ ad } e_{\pm\alpha_0} = 0$

(vi) For the type VII vertex :

$2(\text{ ad } e_{\pm\alpha_2})(\text{ ad } e_{\pm\alpha_1})(\text{ ad } e_{\pm\alpha_3})e_{\pm\alpha_2}$

$(\text{ ad } e_{\pm\alpha_1}) \text{ ad } e_{\pm\alpha_0} - 3(\text{ ad } (e_{\pm\alpha_1})(\text{ ad } e_{\pm\alpha_2})$

$(\text{ ad } e_{\pm\alpha_3})(\text{ ad } e_{\pm\alpha_2})(\text{ ad } e_{\pm\alpha_1}) \text{ ad } e_{\pm\alpha_0} = 0$

For $A(m,n)$ with $m = 0$ or $n = 0$, $F(4)$ and $G(3)$, it is not necessary to impose supplementary conditions. Now we have the matrix a_{ij} which is symmetrizable and indcomposable. An indecomposable Cartan matrix A is that which can not be expressed as $A = DS$, where D is a diagonal matrix and S is a symmetric matrix with entries of D and S are rational numbers. Taking the symmetric matrix S_{ij} as a metric on a root space we get the following identification

$$D_{ij} = \begin{cases} \frac{2}{(\alpha_i,\alpha_j)}\delta_{ij} \text{ where } \alpha_i \text{ is an even simple root i.e., } i \notin \tau \text{ or a nondegenerate} \\ \text{odd root i.,e, } i \in \tau \text{ and } 2\alpha_i \text{ is also a root.} \\ \delta_{ij} \text{ where } \alpha_i \text{ is an degenerate odd root i.,e, } i \in \tau \text{ and } (\alpha_i,\alpha_i) = 0 \end{cases}$$

and $G_{ij} = (\alpha_i, \alpha_j)$.

1.6 Types of Kac-Moody superalgebras

Note - There is no super analog of the Theorem 1.1.5 in terms of classification of Kac-Moody algebra by GCM [7]. Given a class of symmetrizable generalised Cartan matric (GCM) and there associated algebras, we consider three types if superalgebras:

(*i*) Simple Lie superalgebras: These are the finite dimensional Kac-Moody superalgebras.

(*ii*) Affine Kac-Moody superalgebras: These are the sets of infinite dimensional Lie superalgebras and are of two types:

  (*a*) Untwisted affine Kac-Moody superalgebras that corresponds to identity automophisms of corresponding simple Lie superalgebras

  (*b*) Twisted affine Kac-Moody superalgebras correponding to outer automophisms of order 2 or 4 of the corresponding simple Lie superalgebras.

(*iii*) Hyperbolic Kac-Moody superalgebras: This is a subclass of indefinite Kac-Moody superalgebras. Every leading principal submatrix of the GCM of the algebras decomposes into constituents of finite or affine type or equivalently, deletion of a vertex of the Dynkin diagram of finite or affine type.

1.7 Root Systems

Among the simple root systems of linear superalgebra, there exists a simple root system which the number of odd roots is the smallest. Such a simple root system is called the distinguished simple root system. The details root system and simple root systems of basic classical Lie superalgebras in the distinguised basis are given in Table 5. We will first mention the normalizations of the basis of these superalgebras and then the immediate Table gives more details about root systems.

(i) $A(m,n) = \mathfrak{sl}(m+1|n+1)$:

The roots can be expressed in terms of $e_1, \cdots, e_{m+1}, \delta_1, \cdots, \delta_{n+1}$.

The normalizations of the basis on the root systems are given by

$$(e_i, e_j) = \delta_{ij}(i,j = 1, \cdots, m+1), \ \ (\delta_k, \delta_l) = -\delta_{kl}(k,l = 1, \cdots, n+1), \ \ (e_i, \delta_k) = 0$$

The even roots are $\triangle_0 = \{e_i - e_j(i,j = 1,2,\cdots, m+1); \delta_i - \delta_j(i,j = 1,2,\cdots, n+1)$. and the odd roots are $\triangle_1 = \{\pm(e_i - \delta_j);\} i = 1,\cdots, m+1, j = 1,2,\cdots, n+1$ The distinguished positive simple root system

$$\Pi = e_i - e_{i+1}(i \neq m+1), \ \ \delta_{j-1} - \delta_j(1 < j \leq n), \ \ e_{m+1} - \delta_1.$$

(ii) $B(m,n) = osp(m|2n)$:

The roots can be expressed in terms of $e_1, \cdots, e_m, \delta_1, \cdots, \delta_n$.. The normalizations of the basis on the root systems are given by

$$(e_i, e_j) = -\delta_{ij}(i,j = 1, \cdots, m), \ \ (\delta_k, \delta_l) = \delta_{kl}(k,l = 1, \cdots, n+1), \ \ (e_i, \delta_k) = 0.$$

The even roots are

$$\triangle_0 = \{\pm(e_i \pm e_j); \pm e_i; \pm(\delta_i + \delta_j), \pm 2\delta\}.$$

The odd roots are

$$\triangle_1 = \{\pm(e_i \pm \delta_j); \pm\delta_j\}.$$

The simple root system is given by

$$\Pi = \left\{ \begin{array}{c} \delta_1 - \delta_2, \cdots, \delta_n - e_1, \\ e_1 - e_2, e_2 - e_3, \cdots, e_{m-1} - e_m, e_m \end{array} \right\}.$$

(iii) $B(0,n) = \mathfrak{osp}(1|2n)$:

The normalizations of the basis on the root systems are given by

$$(\delta_k, \delta_l) = \delta_{kl}(k, l = 1, \cdots, n).$$

The even roots are

$$\triangle_0 = \{\pm(\delta_i + \delta_j), \pm 2\delta_i\}(i \neq j; i, j = 1, \cdots, n).$$

The odd roots are

$$\triangle_1 = \{\pm\delta_j,\ i = 1, \cdots, n\}.$$

$$\Pi = \{\delta_1 - \delta_2, \cdots, \delta_{n-1} - \delta_n, \delta_n\}.$$

(iv) $C(n+1) = \mathfrak{osp}(2|2n-2)$:

The normalizations of the basis on the root systems are given by

$$(e, e) = 1,\ \ (\delta_k, \delta_l) = -\delta_{kl}(k, l = 1, \cdots, n+1),\ \ (e, \delta_k) = 0.$$

The even roots are

$$\triangle_0 = \{\pm(e_i \pm e_j) \pm (\delta_i + \delta_j), \pm 2\delta_i\}(i \neq j; i, j = 1, \cdots, n).$$

The odd roots are

$$\triangle_1 = \{\pm(e_i \pm \delta_i,\ i = 1, \cdots, n-1\}.$$

The simple root system is given by

$$\Pi = \{e - \delta_1, \delta_1 - \delta_2, \cdot, \delta_{n-1} - \delta_n, 2\delta_n\}.$$

(v) $D(m, n) = \mathfrak{osp}(2m|2n)$:

The normalizations of the basis on the root systems are given by

$$(e_i, e_j) = -\delta_{ij}(i, j = 1, \cdots, m), \ \ (\delta_k, \delta_l) = \delta_{kl}(k, l = 1, \cdots, n), \ \ (e_i, \delta_k) = 0.$$

The even roots are

$$\triangle_0 = \{\pm(e_i \pm e_j); \pm(\delta_i + \delta_j), \pm 2\delta\}.$$

The odd roots are

$$\triangle_1 = \{\pm(e_i \pm \delta_j); i = 1, \cdots, n;\ j = 1, \cdots, m\}.$$

The simple root system is given by

$$\Pi = \left\{ \begin{array}{c} \delta_1 - \delta_2, \cdot, \delta_n - e_1, \\ e_1 - e_2, e_2 - e_3, \cdots, e_{m-1} - e_m, e_{m-1} + e_m \end{array} \right\}.$$

(vi) $G(3)$:

The basis δ and e_1, e_2, e_3. The normalizations of the basis on the root systems are given by

$$(e_i, e_j) = -3\delta_{ij} + 1 (i, j = 1, 2, 3), \ \ (\delta, \delta) = 2, \ \ (e_i, \delta) = 0.$$

The even roots are

$$\triangle_0 = \{e_i - e_j, \pm 2\delta, \pm e_i, (i, j = 1, 2, 3)\}.$$

The odd roots are

$$\triangle_1 = \{\pm\delta + \pm e_i \pm \delta, (i = 1, 2, 3)\}.$$

The distinguished simple root system is $\Pi = \{\delta + e_1, e_3 - e_2, e_2\}$.

(vii) $F(4)$:

The basis are δ and e_1, e_2, e_3. The normalizations of the basis on the root systems are given by

$$(e_i, e_j) = -2\delta_{ij} + 1 (i, j = 1, 2, 3), \ \ (\delta, \delta) = 6, \ \ (e_i, \delta) = 0.$$

The even roots are

$$\triangle_0 = \{\pm(e_i \pm e_j), \pm\delta, \pm e_i, (i = 1, 2, 3)\}.$$

The odd roots are

$$\triangle_1 = \left\{\pm\delta + \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3)\right\}.$$

The distinguished simple root system is

$$\Pi = \{e_1 - e_2, e_2 - e_3, e_3, \delta + \frac{1}{2}(-e_1 - e_2 - e_3)\}.$$

(viii) $D(2,1;\alpha)$:

The basis are e_1, e_2, e_3. The normalizations of the basis on the root systems are given by

$$(e_1, e_1) = -1(1+\alpha)/2, \quad (e_2, e_2) = 1/2, \quad (e_i, e_j) = 0 \ (i \neq j).$$

The even roots are

$$\triangle_0 = \{\pm 2e_i, i = 1, 2, 3\}.$$

The odd roots are

$$\triangle_1 = \{\pm e_1 \pm e_2 \pm e_3\}.$$

The distinguished simple root system is

$$\Pi = \{e_1 - e_2 - e_3, 2e_2, 2e_3\}.$$

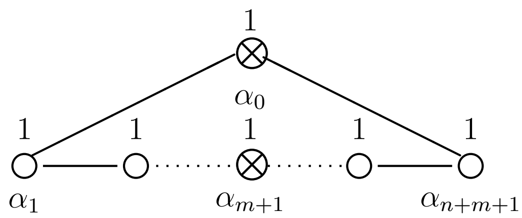
Lie Superalgebra	Dynkin diagram
$A(m,n)$ or $\mathfrak{sl}(m+1,n+1) m,n \geq 0$ $m \neq n$	$\underset{e_1-e_2}{\bigcirc} - \underset{e_2-e_3}{\bigcirc} - \cdots - \underset{e_m-e_{m+1}}{\bigcirc} - \underset{e_{m+1}-\delta_1}{\bigotimes} - \underset{\delta_1-\delta_2}{\bigcirc} - \cdots - \underset{\delta_{n-1}-\delta_n}{\bigcirc} - \underset{\delta_{n-1}-\delta_n}{\bigcirc}$
$B(m,n)$ or $\mathfrak{osp}(2m+1,2n), m \geq 0, n > 0$	$\underset{\delta_1-\delta_2}{\bigcirc} - \underset{\delta_2-\delta_3}{\bigcirc} - \cdots - \underset{\delta_{n-1}-\delta_n}{\bigcirc} - \underset{\delta_n-e_1}{\bigotimes} - \underset{e_1-e_2}{\bigcirc} - \cdots - \underset{e_{m-1}-e_m}{\bigcirc} \Rightarrow \underset{e_m}{\bigcirc}$
$B(0,n)$	$\underset{\delta_1-\delta_2}{\bigcirc} - \underset{\delta_2-\delta_3}{\bigcirc} - \cdots - \underset{\delta_{m-2}-\delta_{m-1}}{\bigcirc} - \underset{\delta_{m-1}-\delta_m}{\bigcirc} \Rightarrow \underset{\delta_m}{\bullet}$
$C(n) = \mathfrak{osp}(2,2n-2), n > 2$	$\underset{e_1-\delta_1}{\bigotimes} - \underset{\delta_1-\delta_2}{\bigcirc} - \cdots - \underset{\delta_{n-2}-\delta_{n-1}}{\bigcirc} - \underset{\delta_{n-1}-\delta_n}{\bigcirc} \Leftarrow \underset{2\delta_n}{\bigcirc}$
$D(m,n)$ or $\mathfrak{osp}(2m,2n), m \geq 2, n \geq 1$	$\underset{\delta_1-\delta_2}{\bigcirc} - \underset{\delta_2-\delta_3}{\bigcirc} - \cdots - \underset{\delta_n-e_1}{\bigotimes} - \bigcirc - \cdots - \bigcirc \begin{matrix} / \bigcirc e_{m-1}-e_m \\ \backslash \bigcirc e_{m-1}+e_m \end{matrix}$
$F(4)$	$\underset{(e_1+e_2+e_3+\delta)/2}{\bigotimes} - \underset{-e_1}{\bigcirc} \Leftarrow \underset{e_1-e_2}{\bigcirc} - \underset{e_2-e_3}{\bigcirc}$
$G(3)$	$\underset{e_3+\delta}{\bigotimes} - \underset{e_1}{\bigcirc} \Lleftarrow - \underset{e_2-e_1}{\bigcirc}$
$D(2,1;\alpha)$	$e_1 - e_2 - e_3 \bigotimes \begin{matrix} / \bigcirc 2e_2 \\ \backslash \bigcirc 2e_3 \end{matrix}$

Table-5

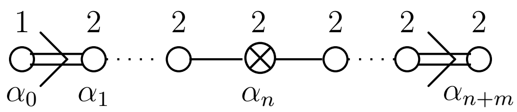
1.8 Dynkin diagrams of affine Kac-Moody superalgebras

The construction of root systems and Dynkin diagrams of affine untwisted Kac-Moody superalgebras run parallel to that of construction of affine Kac-Moody algebras. So without ginving any details we have tabulated the Dynkin diagrams associated with these algebras. For Dynkin diagrams of hyperbolic Kac-Moody superalgebras one can refer [19, 44, 47]. The following graphs are the Dynkin diagram of untwisted affine Kac-Moody superalgebras.

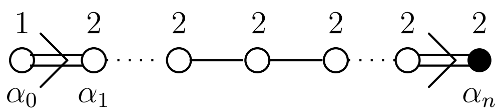
(i) $\mathfrak{sl}(m+1, n+1)^{(1)}$:



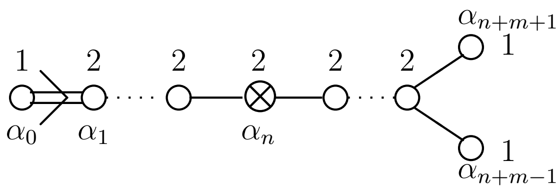
(ii) $\mathfrak{osp}(2m+1, 2n)^{(1)}$:



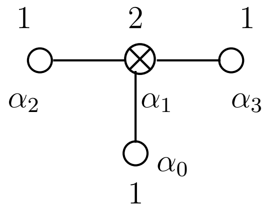
(iii) $\mathfrak{osp}(1, 2n)^{(1)}$:



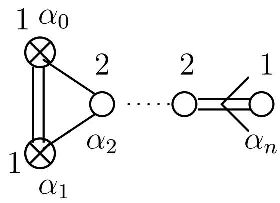
(iv) $\mathfrak{osp}(2m, 2n)^{(1)}$:



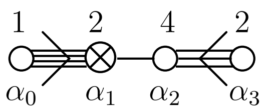
(v) $D(2,1;\alpha)^{(1)}$:



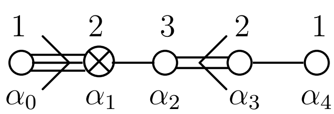
(vi) $\mathfrak{osp}(1,2n-2)^{(1)}$:



(vii) $G(3)^{(1)}$:

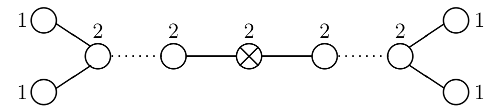


(viii) $F(4)^{(1)}$:

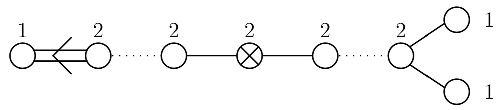


1.8.1 Dynkin diagrams of twisted Kac-Moody superalgebras

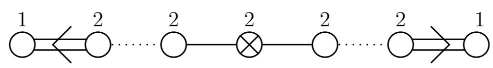
The Dynkin diagram of $\mathfrak{sl}(2m+1|2n)^2$:



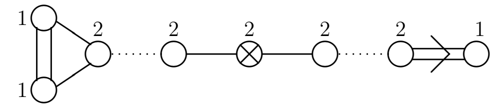
The Dynkin diagram of $\mathfrak{sl}(2m+1|2n)^2$:



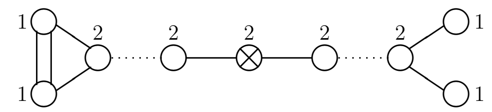
The Dynkin diagram of $\mathfrak{sl}(2m+1|2n+1)^2$:



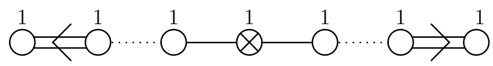
The Dynkin diagram of $\mathfrak{sl}(2|2n+1)^{(2)}$:



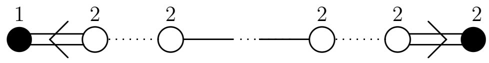
The Dynkin diagram of $\mathfrak{sl}(2|2n)^{(2)}$:



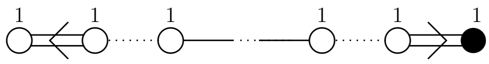
The Dynkin diagram of $\mathfrak{osp}(2m|2n)^{(2)}$:



The Dynkin diagram of $\mathfrak{osp}(2|2n)^{(2)}$:



The Dynkin diagram of $\mathfrak{sl}(1|2n+1)^{4}$:



1.9 Real forms

Let V be a vector space over $\mathbb{R}$ of finite dimension. A complex structure on V is an R-linear endomorphism J of V such that $J^2 = -I$, where I is the identity mapping of V. A vector space V over $\mathbb{R}$ with a complex structure J can be turned into a vector space

For field $\Bbbk = \mathbb{R}$ and $\mathbb{C}$, V is a real vector space, the complex vector space $V^{\mathbb{C}}$ is called the complexification of V . If W is complex, then $W^{\mathbb{R}}$ is regarded as a real vector space. The operations $(\centerdot)^{\mathbb{C}}$ and $(\centerdot)^{\mathbb{R}}$ are not inverse to each other: $(V^{\mathbb{C}})^{\mathbb{R}}$ has twice the real dimension of V and $(W^{\mathbb{R}})^{\mathbb{C}}$ has twice the real dimension of W. More precisely

$$(V^{\mathbb{C}})^{\mathbb{R}} = V \oplus iV$$

Definition 1.9.1. ([36] Knapp A W) When a complex vector space W and a real vector space V are related by

$$W^{\mathbb{R}} = V \oplus iV$$

Then we say that V is a real form of the complex vector space W.

Definition 1.9.2. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. A real form of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of the real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ such that $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus J\mathfrak{g}_0$. ($\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{g}_0$).

1.10 Split form and compact form

Every semisimple Lie algebra possesses two particular real forms in practice i.e, the split form and the compact form. These represent the two extremes behavior of the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha)$ with respect to real subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$. The other real forms of an algebra lie in between these two extremes.

The split form of $\mathfrak{g}$ is a form $\mathfrak{g}_0$ such that there exists a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}$ whose action on $\mathfrak{g}_0$ has all real eigenvalues, i.e., all the roots $\alpha \in R \subset \mathfrak{h}^*$ of $\mathfrak{g}$ (with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}$) assume all real values on the subspace $\mathfrak{h}_0$ and we have a direct sum decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus (\bigoplus \mathfrak{l}_\alpha)$$

of $\mathfrak{g}_0$ into $\mathfrak{h}_0$ and the one dimensional eigenspaces $\mathfrak{l}_\alpha$ for the action of $\mathfrak{h}_0$ (each $\mathfrak{l}_\alpha$ will just the intersection of the root space $\mathfrak{g}_\alpha \subset \mathfrak{g}$); each pair $\mathfrak{l}_\alpha$ and $\mathfrak{l}_{-\alpha}$ will generate a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

On the other hand, in the compact form all the roots $\alpha \in R \subset \mathfrak{h}^*$ of $\mathfrak{g}$ assume all purely imaginary values on the subspaces $\mathfrak{h}_0$. Then we have a direct sum decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus (\bigoplus \mathfrak{l}_\alpha)$$

of $\mathfrak{g}_0$ into $\mathfrak{h}_0$ and the two dimensional eigenspaces $\mathfrak{l}_\alpha$ (each $\mathfrak{l}_\alpha$ will just the intersection of the root space $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\alpha$ with $\mathfrak{g}_0$); each $\mathfrak{l}_\alpha$ will generate a subalgebra isomorphic to $\mathfrak{su}(2)$.

Proposition 1.10.1 ([36], Theorem 6.94). *Let* $\mathfrak{g}_0$ *be a simple Lie algebra over* $\mathbb{R}$ *and* $\mathfrak{g}$ *be its complexification. Then there are just two possibilies:*

(i) $\mathfrak{g}_0$ *is complex, i.e., is of the form* $\mathfrak{s}^{\mathbb{R}}$ *for some complex* $\mathfrak{s}$ *and then* $\mathfrak{g}$ *is* $\mathfrak{s}^{\mathbb{C}}$ *isomorphic to* $\mathfrak{s} \oplus \mathfrak{s}$

(ii) $\mathfrak{g}_0$ *is not complex and then* $\mathfrak{g}$ *is simple over* $\mathbb{C}$*.*

Proposition 1.10.2 ([36], Proposition 6.95). *If* $\mathfrak{g}$ *is a complex Lie algebra simple over* $\mathbb{C}$*, then* $\mathfrak{g}^{\mathbb{R}}$ *is simple over* $\mathbb{R}$*.*

Proposition 1.10.3 ([29], Proposition 7.4). *Let* $\mathfrak{g}_0$ *be a semisimple Lie algebra over* $\mathbb{R}$ *which is the direct sum* $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$*, where* $\mathfrak{k}_0$ *is a subalgebra and* $\mathfrak{p}_0$ *is a vector subspace. The following are equivalent*

(i) $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ *is a Cartan decomposition of* $\mathfrak{g}_0$*.*

(ii) The mapping $s_0 : T + X \to T - X \ \ (T \in \mathfrak{k}_0, X \in \mathfrak{p}_0)$ *is an automorphism of* $\mathfrak{g}_0$ *and the symmetric bilinear form*

$$B_{s_0}(X, Y) = -B(X, s_0 Y)$$

is strictly positive definite (that is, $B < 0$ *on* $\mathfrak{k}_0$*,* $B > 0$ *on* $\mathfrak{p}_0$ *)*

If these conditions are satisfied, $\mathfrak{k}_0$ *is a maximally compactly imbedded subalgebra of* $\mathfrak{g}_0$*.*

Theorem 1.10.4 ([36], Theorem 6.105). *Up to isomorphism every simple real Lie algebra is in the following list and everything in the list is a simple real Lie algebra:*

(a) the Lie algebra $\mathfrak{g}^{\mathbb{R}}$*, where* $\mathfrak{g}$ *is complex simple of type* A_n *for* $n \geq 1$*.*

(b) the compact real form of any $\mathfrak{g}$ *as in a,*

(c) the classical matrix algebras

$$\begin{array}{llll}
\mathfrak{su}(p,q) & \textit{with} & p \geq q > 0, & p+q \geq 2 \\
\mathfrak{so}(p,q) & \textit{with} & p > q > 0, & p+q \textit{ odd} \quad p+q \geq 5 \\
 & \textit{or with} & p \geq q > 0, & p+q \textit{ even} \quad p+q \geq 8 \\
\mathfrak{sp}(p,q) & \textit{with} & p \geq q > 0, & \\
\mathfrak{sp}(n,\mathbb{R}) & \textit{with} & n \geq 3 & \\
\mathfrak{so}^*(2n) & \textit{with} & n \geq 4 & \\
\mathfrak{sl}(n,\mathbb{R}) & \textit{with} & n \geq 3 & \\
\mathfrak{sl}(n,\mathbb{H}) & \textit{with} & n \geq 2 &
\end{array}$$

(d) the 12 exceptional noncomplex noncompact simple Lie algebras given in Table 8 and 9.

Theorem 1.10.5 ([36], Theorem 6.11)**.** *If* $\mathfrak{g}$ *is a complex semisimple Lie algebra, then* $\mathfrak{g}$ *has a compact real form* $\mathfrak{u}_0$*.*

Let the conjugate transpose mapping $X \to X^*$ or involution is given by $\theta(X) = -X^*$. θ preserve the brackets since $\theta[X,Y] = -[X,Y]^* = -[Y^*,X^*] = [-X^*,-Y^*] = [\theta(X),\theta(Y)]$

Proposition 1.10.6 ([36], Proposition 1.119)**.** *If* B *is the Killing form of* $\mathfrak{g}$ *and* a *is an automophism of* $\mathfrak{g}$*, then* $B(aX,aY) = B(X,Y)$ *for all* X *and* Y *in* $\mathfrak{g}$*.*

The involution θ has the property that $B(\theta X,\theta Y) = -B(X,\theta Y) = -B(\theta X,\theta^2 Y) = -B(\theta X,Y) = -B(Y,\theta X) = B_\theta(Y,X)$ which gives $B_\theta(X,X) \geq 0$.

Proposition 1.10.7 ([36], Proposition 7.17)**.** *Let* $\mathfrak{g}_0$ *be a real semisimple Lie algebra,* θ *be an involution of* $\mathfrak{g}_0$ *and* B *be a nondegenerate symmetric invariant bilinear form on* $\mathfrak{g}_0$ *such that* $B(\theta X,\theta Y) = B(X,Y)$ *for all* X *and* Y *in* $\mathfrak{g}_0$*. If the* *form* $B_\theta(X,Y) = -B(X,\theta Y)$ *is positive definite, then* θ *is a Cartan involution of* $\mathfrak{g}_0$.

1.11 Vogan diagram

Although, the real forms of algebras have already been classified, the theory of Vogan diagrams introduces invariants for such algebras; which makes it possible to locate a given real forms within the classification. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, $\mathfrak{g}$ be its complexification and θ be a Cartan involution. Then $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the corresponding Cartan decomposition. A θ stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0$ is maximally compact if its compact dimension is as large as possible, maximally noncompact if its noncompact dimensional is as large as possible. We let a maximally compact θ stable Cartan subalgebra $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ of $\mathfrak{g}_0$ with complexification $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ and $\triangle = \triangle(\mathfrak{g}, \mathfrak{h})$ be the set of roots. Choose a positive system $\triangle^+$ for $\triangle$ that takes $i\mathfrak{t}_0$ before $\mathfrak{a}$. For example, $\triangle^+$ can be defined in terms of lexicographic ordering built from a basis of $i\mathfrak{t}_0$ followed by a basis of $\mathfrak{a}_0$. $\theta(\mathfrak{h}_0) = \mathfrak{t}_0 \oplus (-1)\mathfrak{a}_0$, hence θ is $+1$ on $\mathfrak{t}_0$ and -1 on $\mathfrak{a}_0$ and there are no real roots, $\theta(\triangle^+) = \triangle^+$. Therefore θ permutes the simple roots. It must fix the simple roots that are imaginary and permute in 2-cycles the simple roots that are complex. By the Vogan diagram of the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \triangle^+)$., we mean the Dynkin diagram of $\triangle^+$ with the 2 element orbits under θ so labeled and with the 1-element orbits painted or not, according as the corresponding imaginary simple root is noncompact or compact. An important result in the theory of Vogan diagrams for real simple Lie algebras states that any Vogan diagram can be transformed, by changing the ordering of its base, into a diagram which has at most one non- compact imaginary root and that root occurs at most twice in the largest root of that simple Lie algebra. Since in the case of affine Kac-Moody algebras, changing the order does not give a Vogan diagram with at most one shaded root, therefore a notion of equivalence of Vogan diagrams for non-twisted affine Kac-Moody Lie algebras was introduced by Batra in [2,3]. Tanushree [48] modified the definition of the Vogan diagrams for the twisted affine Kac-Moody Lie algebras.

Definition 1.11.1. [36] We define an abstract Vogan diagram to be an abstract Dynkin diagram with two pieces of additional strucuture indicated: One is an automophism of order 1 or 2 of the diagram, which is to be indicated by labelling the 2-element orbits. The other is a subset of the 1-element orbits, which is to be indicated by painting the vertices correspoding to the members of the subset.

Remark 1.11.2. Every Vogan diagram is of course an abstract Vogan diagram.

Theorem 1.11.3. *[36] If an abstract Vogan diagram is given, then there exist a real semisimple Lie algebra* ($\mathfrak{g}_0$*, a Cartan involution* θ*, a maximally compact* θ *stable Cartan subalgebra* $\mathfrak{h}_0) = \mathfrak{t}_0 \oplus (-1)\mathfrak{a}_0$*, and a positive system* $\triangle^+$ *for* $\triangle = \triangle(\mathfrak{g}, \mathfrak{h})$ *that takes* $i\mathfrak{t}_0$ *before* $\mathfrak{a}$ *such that the given diagram is the Vogan diagram of* $(\mathfrak{g}_0, \mathfrak{h}_0, \triangle^+)$.

Remark 1.11.4. The theorem says that any abstract Vogan diagrams comes from some $\mathfrak{g}_0$.

Lemma 1.11.5 ([36],Lemma 6.97)**.** *Let* $\triangle$ *be an irreducible abstract reduced root system in a real vector superspace* V*, let* Π *be a simple simple root system and let* ω *and* ω' *be nonzero members of* V *that are domiant relative to* Π_i'*s. Then* $\langle \omega, \omega' \rangle > 0$.

Lemma 1.11.6 ([36], Lemma 6.98)**.** *Let* $\mathfrak{g}_0$ *be a noncomplex simple real Lie algebra and the Vogan diagram of* $\mathfrak{g}_0$ *be given that corresponds to the triple* $(\mathfrak{g}_0, \mathfrak{h}_0, \triangle^+)$*. Write* $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$*. Let* V *be the span of the simple roots that are imaginary,* $\triangle_0$

be the system $\triangle \cap V$, $\mathcal{H}$ *be the subsets of* $i\mathfrak{t}_0$ *paired with* V *and let* Λ *be the subset of* $\mathcal{H}$ *where all roots of* $\triangle_0$ *take integer values and all noncompact roots of* $\triangle_0$ *take odd integer values. Then* Λ *is nonempty. In fact* $\alpha_1 \cdots, \alpha_m$ *is any simple system for* $\triangle_0$ *and if* $\omega_1, \cdots, \omega_m$ *in* V *are defined by* $\langle \omega, \alpha_k \rangle = \delta_{jk}$, *then the element*

$$\omega = \sum_{i \text{ with } \alpha_i \text{ noncompact}} \omega_i$$

Theorem 1.11.7 ([36], Theorem 6.96, Borel and de Siebenthal Theorem)**.** *Let* $\mathfrak{g}_0$ *be a non complex simple real Lie algebra, and let the Vogan diagram of* $\mathfrak{g}_0$ *be given that corresponds to the triple* $(\mathfrak{g}_o, \mathfrak{h}_o, \triangle^+)$. *Then there exists a simple system* $\prod'$ *for* $\triangle = \triangle(\mathfrak{g}, \mathfrak{h})$, *with corresponding positive system* $\triangle^{+'}$, *such that* $(\mathfrak{g}_o, \mathfrak{h}_o, \triangle^+)$ *is a triple and there is at most one painted simple root in its Vogan diagram. Furthermore suppose that automorphism associated with the Vogan diagram is the identity, that* $\prod' = \{\alpha_1, \cdots, \alpha_l\}$ *and that* $\{\omega_1, \cdots, \omega_l\}$ *is the dual basis given by* $< \omega_j, \alpha_k >= \delta_{jk}$. *Then the single painted simple root* α_i *may be choosen so that there is no* i' *with* $\langle \omega_i - \omega_j, \omega_{i'} \rangle > 0$.

Theorem 1.11.8 ([36], Theorem 6.74)**.** *Let* $\mathfrak{g}_0$ *and* $\mathfrak{g}_0'$ *be real semisimple Lie algebras. If two triples* $(\mathfrak{g}_0, \mathfrak{h}_0, \triangle^+)$ *and* $(\mathfrak{g}_0', \mathfrak{h}_0', (\triangle')^+)$ *have the same Vogan diagram, then* $\mathfrak{g}_0$ *and* $\mathfrak{g}_0'$ *are isomorphic.*

Now we list the Vogan diagrams of simple Lie algebras and their associated real forms in Table 6,7,8,9. Similarly the Vogan diagrams of affine untwisted Kac-Moody algebras have been determined by Punita Batra [2] and by Tanushree Pal for twisted case [48]. In our thesis we have extended these theory to hyperbolic Kac-Moody algebras in the next Chapter.

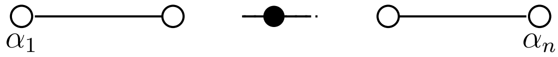
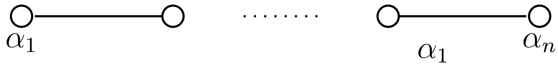
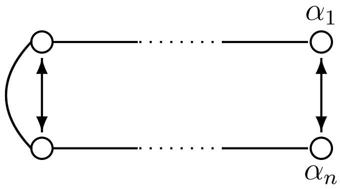
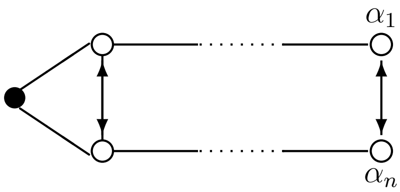
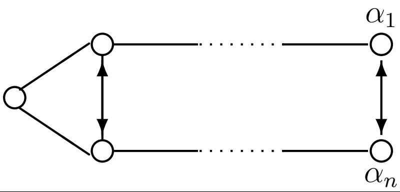
Lie algebra	Vogan diagram	Real form
A_n		$\mathfrak{su}(\mathfrak{p},\mathfrak{q})$
		$\mathfrak{su}(\mathfrak{n})$
		$\mathfrak{sl}(n+1,\mathbb{R})$
		$\mathfrak{sl}(n+1,\mathbb{R})$
		$\mathfrak{sl}_{\frac{1}{2}}(n+1,\mathbb{R})$

Table - 6

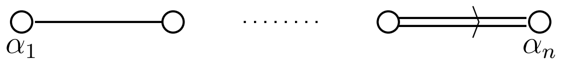
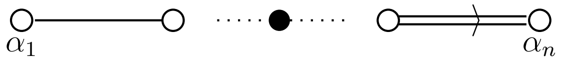
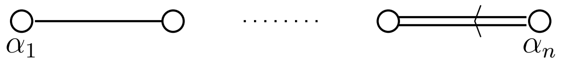
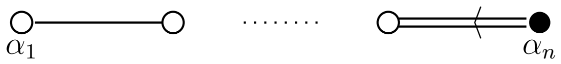
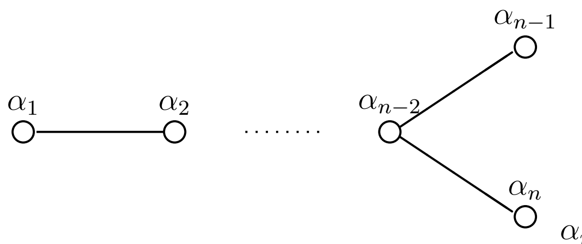
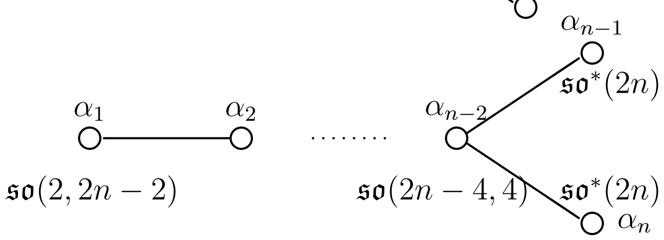
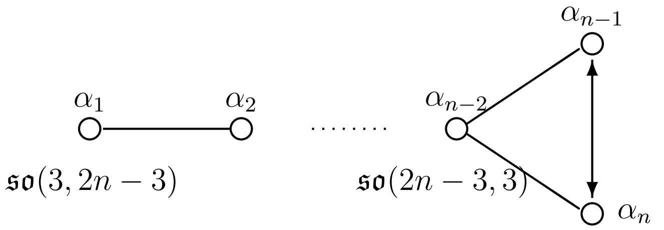
Lie algebra	Vogan diagram	Real form
B_n		$\mathfrak{so}(2n+1)$
		$\mathfrak{so}(p,q)$
C_n		$\mathfrak{sp}(p,q)$
		$\mathfrak{sp}(n,\mathbb{R})$
D_n		$\mathfrak{so}(2n)$
		
		

Table - 7

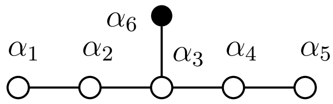
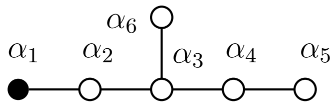
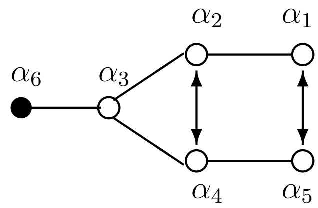
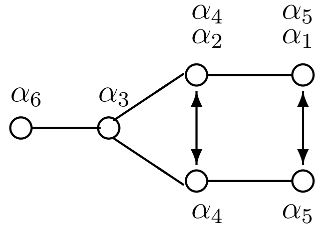
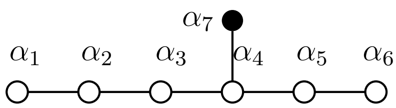
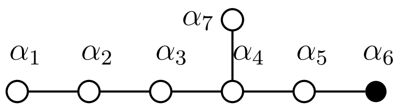
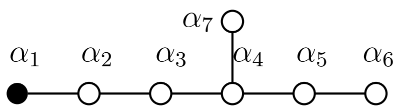
Lie algebra	Vogan diagram	Real form
E_6	 $\mathfrak{k}_0 = \mathfrak{su}(6) \oplus \mathfrak{su}(2)$	E II
	 $\mathfrak{k}_0 = \mathfrak{so}(10) \oplus \mathbb{R}$	E III
		E I
		E IV
E_7	 $\mathfrak{k}_0 = \mathfrak{su}(8)$	E V
	 $\mathfrak{k}_0 = \mathfrak{so}(12) \oplus \mathfrak{su}(2)$	E VI
	 $\mathfrak{k}_0 = \mathfrak{e}_6 \oplus \mathbb{R}$	E VII

Table - 8

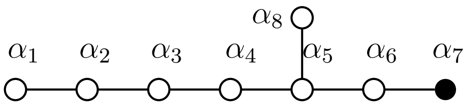
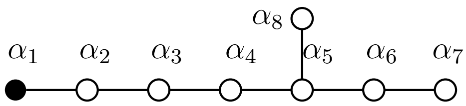
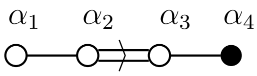
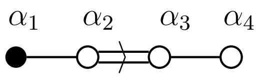
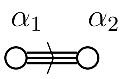
Lie algebra	Vogan diagram	Real form
E_8	 $\mathfrak{k}_0 = \mathfrak{so}(16)$	E VII
	 $\mathfrak{k}_0 = \mathfrak{e}_7 \oplus \mathfrak{su}(2)$	E IX
F_4	 $\mathfrak{k}_0 = \mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	F I
	 $\mathfrak{k}_0 = \mathfrak{so}(9)$	F II
G_2	 $\mathfrak{k}_0 = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$	G

Table - 9

Chapter 2

Vogan diagrams of hyperbolic Kac-Moody algebras

In this chapter we extend the theory of the Vogan diagrams to that of hyperbolic Kac-Moody algebras. It is now believed that the classification of hyperbolic Kac-Moody algebras is complete [43, 54, 55]. The list is very large. In principle one can construct Vogan diagrams of all hyperbolic Kac-Moody algebras. However it is a huge and daunting task. For this reason we have restricted ourselves to those algebras which have some potential physical applications. We believe such type of studies of hyperbolic Kac-Moody algebras by Vogan diagrams will be helpfull to solve many problems related to string theory and oxidation of sigma model in association with Magic triangle and M-theory [50]. From algebraic point of view this also motivates to constuct and develop the Vogan diagram of other hyperbolic Kac-Moody algebras.

2.1 Real form of hyperbolic Kac-Moody algebras and Vogan diagram

A real form of $\mathfrak{g}$ is an algebra $\mathfrak{g}_{\mathbb{R}}$ over $\mathbb{R}$ such that there exists an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$. If we replace $\mathbb{C}$ with $\mathbb{R}$ in the definition of $\mathfrak{g}$, we obtain a real form $\mathfrak{g}_{\mathbb{R}}$ which is called split. A real form of $\mathfrak{g}$ correponds to a semi-linear involution of $\mathfrak{g}$. A semi-linear involution is an automorphism τ of $\mathfrak{g}$ such that $\tau^2 = Id$ and $\tau(\lambda x = \overline{\lambda}\tau(x))$ for $\lambda \in \mathbb{C}$.

A Borel subalgebra of $\mathfrak{g}$ is a maximal completely solvable subalgebra. There are two types of Borel subalgebra (positive or negative). A linear or semi-linear automorphism of $\mathfrak{g}$ is said to be of the first kind if it transforms a Borel subalgebra in $\mathfrak{g}$ to a Borel subalgebra of the same sign. A linear or semilinear automorphism of $\mathfrak{g}$ is said to be of the second kind if it transforms a Borel subalgebra into a Borel subalgebra of the oppposite sign.

Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of $\mathfrak{g}$ and fix an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{R}\otimes\mathbb{C}}$. Then the Galois group $\Gamma = Gal(\mathbb{C}/\mathbb{R})$ acts on $\mathfrak{g}$ and the correponding group G. A real form is the fixed point set $\mathfrak{g}^{\Gamma}$. If Γ consists of first kind of automorphism then $\mathfrak{g}_{\mathbb{R}}$ is almost split, otherwise if the non-trivial element of Γ is a second kind automorphism the $\mathfrak{g}_{\mathbb{R}}$ is almost compact.
We consider

1. The semi-involutions σ', of the second kind of $\mathfrak{g}$ and σ a Cartan involution.

2. The involution θ, of the first kind of $\mathfrak{g}$.

Theorem 2.1.1. *[52] The relation $\sigma \approx \theta$ if and only if*

1. $\omega' = \theta\sigma' = \sigma'\theta$ is a Cartan semi-involution.

2. θ and σ' stabilize the same Cartan subalgebra $\mathfrak{h}$.

3. $\mathfrak{h}$ *is contained in a minimal* σ'*-stable positive parabolic subalgebra.*

Remark 2.1.2. The above theorem is proved in affine Kac-Moody algebras. Since the minimal parabolic subalgebra $\mathfrak{p}$, satisfies $\sigma'(\mathfrak{p}) \cap \mathfrak{p} = \mathfrak{h}$. The condition in the theorem is equivalent for hyperbolic Kac-Moody algebra with $\mathfrak{h}^{\sigma'}$ is a maximally compact Cartan subalgebra. Also this relation induces a bijection between the conjugacy classes under $Aut(\mathfrak{g})$ of semi-involutions of the second kind and conjugacy classes of involutions of the first kind.

2.2 Vogan diagram

Let $\mathfrak{g}_{\mathbb{R}}$ be an almost compact real form of $\mathfrak{g}$, σ be the Cartan involution and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the correspoding Cartan decomposition. Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be a maximally compact σ-stable Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$, with complexification $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$

The roots of $(\mathfrak{g}, \mathfrak{h})$ are imaginary on $\mathfrak{t}_0$ and real on $\mathfrak{a}_0$. A root is real if it takes real values on $\mathfrak{h}_0$ (i.e., vanishes on $\mathfrak{t}_0$), imaginary if it takes purely imaginary values on $\mathfrak{h}_0$ (i.e., vanishes on $\mathfrak{a}_0$) and complex otherwise.

Let $\triangle = \triangle(\mathfrak{g}, \mathfrak{h})$ be the set of roots. There are no real roots, i.e., no roots that vanishes on $\mathfrak{t}$.

We choose a positive system $\triangle^+$ for $\triangle$ that takes $i\mathfrak{t}_0$ before $\mathfrak{a}_0$. Since σ is $+1$ on $\mathfrak{t}_0$ and -1 on $\mathfrak{a}_0$ and since there are no real roots, $\sigma(\triangle^+) = \triangle^+$. Therefore σ permutes the simple roots. It must fix the simple roots that are imaginary and permutes in two cycles the simple roots that are complex.

By the Vogan diagram of the triple $(\mathfrak{g}, \mathfrak{h}_0, \triangle^+)$, we mean the Dynkin diagram of $\triangle^+$ with the two-element orbits under σ (*diagram automorphism*) labelled by an arrow and with the one-element orbits painted or not, according to the correponding imaginary simple root, noncompact or compact.

A combinatorial description for two Vogan diagrams to be equivalent with $F\langle i\rangle$ operation is as follows [12]

(i) The color of i and all vertices not adjacent to i remain unchanged.

(ii) If j is joined to i by a double edge and j 's long, the color of j remains unchanged.

(iii) Apart from the above exceptions, the colors of all vertices adjacent to i, are reversed.

Here we reformulate the Borel and de Siebental Theorem for hyperbolic Kac-Moody algebras.

Theorem 2.2.1. *(Borel and de Siebenthal) Every equivalence class of very extended (hyperbolic) Vogan diagrams has a representative with at most one vertex painted.*

Proof. The Borel and de Siebenthal theorem says that every real form of a complex simple Lie algebra can be represented by a Vogan diagram with at most one painted vertex. This was verified by using algorithm $F\langle i\rangle$ in [12] and diagram automorphisms to explicitly reduce every painting on a Dynkin diagram D to another painting with at most one painted vertex.

In [10], it is shown that the equivalence class of Vogan diagram of an extended Dynkin diagram D, with an extra vertex p with extra edge consists of at most two painted vertices.

The hyperbolic Dynkin diagram consists of a very extra vertex with very extra edge with at most two painted vertices, we obtain a painting with at most three painted vertices. But using the same $F\langle i\rangle$ algorithm we get a painting with at most one painted vertex. $\square$

The following result is immediate.

Corollary 2.2.2. *If a connected graph D is a hyperbolic Dynkin diagram, then*

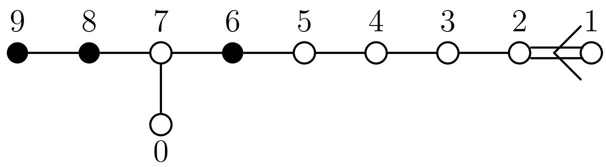
(a) every painting on D can be simplified via a sequence of $F\langle i\rangle$ to a painting with single painted vertex.

(b) every connected subgraph of D satisfies property (a).

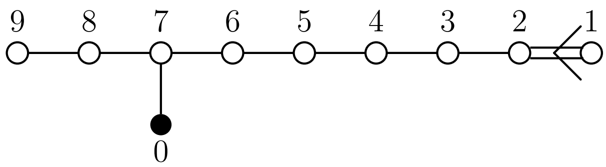
Proof. The proof of the corollary is straightforwad using the $F\langle i\rangle$ sequences. $\square$

The Vogan diagrams $HA^{(2)}_{15,8}$ is a result of Theorem 2.2.1 and Corollary 2.2.2

Using the Theorem 2.2.1 without using the reduction by the $F\langle i\rangle$ algorithm, we get at most three painted vertices Vogan diagrams.



The above Vogan diagram with vertices painted $(9,8,6)$ is equivalent to single painted Vogan diagram, by using Theorem 2.2.2 and hence we get $(9,8,6)\sim(8,7)\sim(0,7)\sim(0)$ by $F\langle 8,7,0\rangle$. So the equivalent diagram becomes



2.3 Equivalence classes of Vogan diagrams

The equivalence of Vogan diagrams is defined by the equivalence relations generated by the following two operations:

(a) Application of an automorphism of the Dynkin diagram.

(b) Change in the positive system by reflection in a simple, noncompact root, i.e., by a vertex which is colored in the Vogan diagram.

As a consequence of reflection by a simple, noncompact root α, the rule for single and triple lines is that we leave α colored and its immediate neighbour is changed to the opposite color. The rule for double lines is that if α is the smaller root, then there is no change in the color of its immediate neighbour, but we leave α colored. If α is the larger root, then we leave α colored and its immediate neighbour is changed to the opposite color. If two Vogan diagrams are not equivalent to each other, they are called nonequivalent.

By labeling vertices with $1, \cdots, n$ as in [11, 12] a Vogan diagram with painted vertices $i_1, \cdots, i_k$, $1 \leq i_1 \leq \cdots < i_k \leq n$, is denoted by $(i_1, \cdots, i_k)$. Suppose $i \in \{i_1, \cdots, i_k\}$, so that i is a painted vertices. The $F\langle i \rangle$ operates on the Vogan diagram as follows. It acts on the root system by reflection corresponding to the noncompact simple root i, as a result it leads an equivalent Vogan diagram.

For Exceptional Dynkin diagram like E_{10}, we get the following Lemma from [10]

Lemma 2.3.1 ([10], Lemma 3.8). *For the integers p, q the follwing holds.*

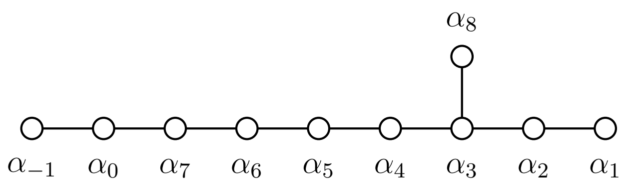
(a) For $q \geq 4$ and $p = 2, 3$, we get $(p, q) \sim (0, p-1, q-1)$ and $(0, p, q) \sim (p-1, q-1)$

(b) For $q \geq 4$, $(1, q) \sim (0, q-1)$ and $(0, 1, q) \sim (q-1)$

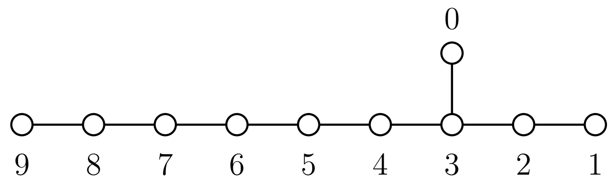
Applying all these available techniques, we proceed to find the Vogan diagrams of some algebras which are important in respect of physical applications in string theory, high eneregy physics etc,. We start with E_{10}.

Example 2.3.2. Dynkin diagram of hyperbolic Kac-Moody algebra E_{10}.

The Dynkin diagram of E_{10} is given as:



α_0 and α_{-1} roots correspond to affine and hyperbolic extension of simple Lie algebra E_8 respectively. Enumerating the vertices of E_{10} differently as shown below we get the Dynkin diagram and the following results.



Proposition 2.3.3. *The equivalence classes of Vogan diagram of E_{10} are given by*

1. $1 \sim 5 \sim (0,4) \sim (0,9) \sim (0,8)$

2. $2 \sim 3 \sim 7 \sim 8 \sim (0,7) \sim (0,6) \sim 0 \sim 4 \sim 6 \sim 9 \sim (0,3) \sim (0,1) \sim (0,2) \sim (0,5) \sim (0,7)$

Proof. The proposition can be proved by using Lemma 2.6.1, switching the sequences as follows:

$$
\begin{aligned}
(0,7) &\sim (1,8) && \text{by Lemma 2.6.1(b)}\\
&\sim (0,2,9) && \text{by Lemma 2.6.1(a)}\\
&\sim (8) && F\langle 0,3,4,5,6,7,8\rangle\\
&\sim (0,1,9) && \text{by Lemma 2.6.1(b)}\\
&\sim (2) && \text{by} F\langle 9,8,7,6,5,4,3,2\rangle\\
&\sim (0,1,4) && \text{by F}\langle 2,3,0\rangle\\
&\sim (3) && \text{by Lemma 2.6.1(b)}
\end{aligned}
$$

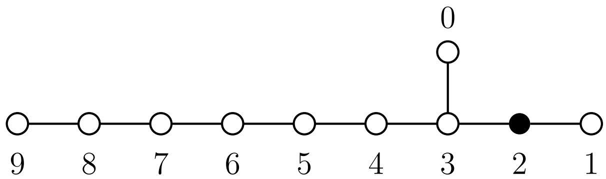
$$
\begin{aligned}
(0,5) &\sim (1,6) && \text{by Lemma 2.6.1(b)}\\
&\sim (0,2,7) && \text{by Lemma 2.6.1(a)}\\
&\sim (6) && \text{by} F\langle 0,3,4,5,6\rangle\\
&\sim (0,1,7) && \text{by Lemma 2.6.1(b)}\\
&\sim (2,8) && \text{by Lemma 2.6.1(a)}\\
&\sim (0) && F\langle 8,7,6,5,4,3,0\rangle\\
&\sim (0,3) && \text{F}\langle 0\rangle\\
&\sim (1,4) && \text{by Lemma 2.6.1(b)}\\
&\sim (0,2,5) \sim (4) && F\langle 0,3,4\rangle\\
&\sim (0,1,5) && \text{by Lemma 2.6.1(b)}\\
&\sim (2,6) && \text{by Lemma 2.6.1(b)}\\
&\sim (0,7) && F\langle 6,5,4,3,0\rangle\\
&\sim (1,8) && \text{by Lemma 2.6.1(b)}\\
&\sim (0,2) && F\langle 8,7,6,5,4,3,2\rangle\\
&\sim (0,1) && F\langle 0,2,1\rangle\\
&\sim (9) && \text{by F}\langle 1,2,3,4,5,6,7,8,9\rangle
\end{aligned}
$$

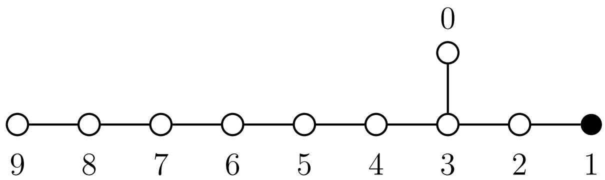
$$
\begin{aligned}
(5) \sim\ & (0,1,6) && \text{by Lemma 2.6.1(b)}\\
\sim\ & (1,5) && F\langle 0,1,3,4,5\rangle\\
\sim\ & (0,4) && \text{by Lemma 2.6.1(b)} \sim (1) \text{ by } F\langle 0,3,2,1\rangle\\
\sim\ & (0,9) && F\langle 1,2,3,4,5,6,7,8,9\rangle
\end{aligned}
$$

$$
\begin{aligned}
(0,6) \sim\ & (1,7) && \text{by Lemma 2.6.1(b)}\\
\sim\ & (0,2,8) && \text{by Lemma 2.6.1(a)}\\
\sim\ & (7) && \text{by} F\langle 0,3,4,5,6\rangle\\
\sim\ & (0,1,8) && \text{by Lemma 2.6.1(b)}\\
\sim\ & (2) && \text{by F}\langle 8,7,6,5,4,3,2\rangle\\
\sim\ & (0,1,4) && \text{by F}\langle 2,3,0\rangle\\
\sim\ & (3) && \text{by Lemma 2.6.1(b)}
\end{aligned}
$$

□

Remark 2.3.4. The two Vogan diagrams of hyperbolic Kac-Moody algebra E_{10} are given below.

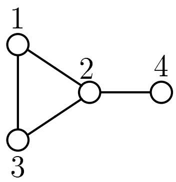




Now we consider some more examples.

Example 2.3.5. Dynkin diagram of a rank 4 hyperbolic algebra:

Consider the Dynkin diagram



Proposition 2.3.6. *The equivalences classes of Vogan diagram of the above hyperbolic Kac-Moody algebra are given by*

(a) $1 \sim 3 \sim 4$

(b) 2

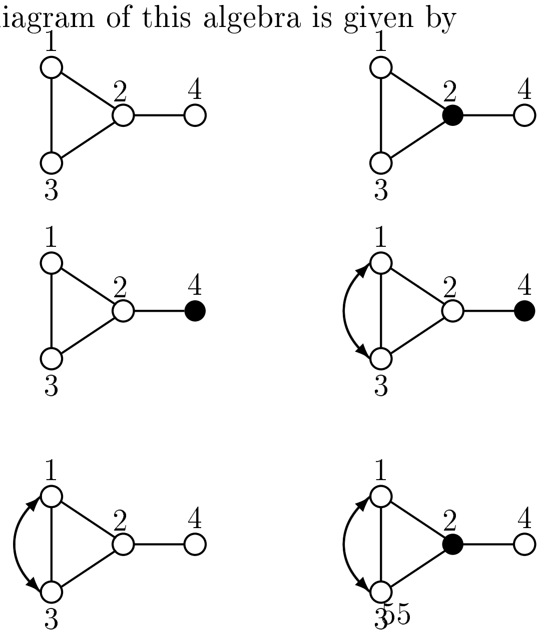
Proof. By applying Lemma 2.6.1 [10]

$$\begin{aligned}
(1) &\sim (1,2,3) && \text{by } F\langle 1\rangle \\
&\sim (2,4) && \text{by } F\langle 2\rangle \\
&\sim (4) && \text{by } F\langle 4\rangle \\
(1,2,3) &\sim (3) && \text{by } F\langle 3\rangle
\end{aligned}$$

Also we have

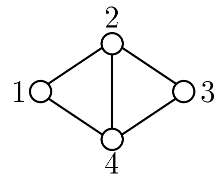
$$\begin{aligned}
(2) &\sim (1,2,3,4) && F\,\langle 2\rangle \\
&\sim (1,3,4) && F\,\langle 4\rangle
\end{aligned}$$

Vogan diagram of this algebra is given by



□

Example 2.3.7. The Dynkin digram of another rank 4 hyperbolic Kac-Moody algebra is given by



Proposition 2.3.8. *The equivalence classes of another rank 4 hyperbolic Kac-Moody algebra are given by*

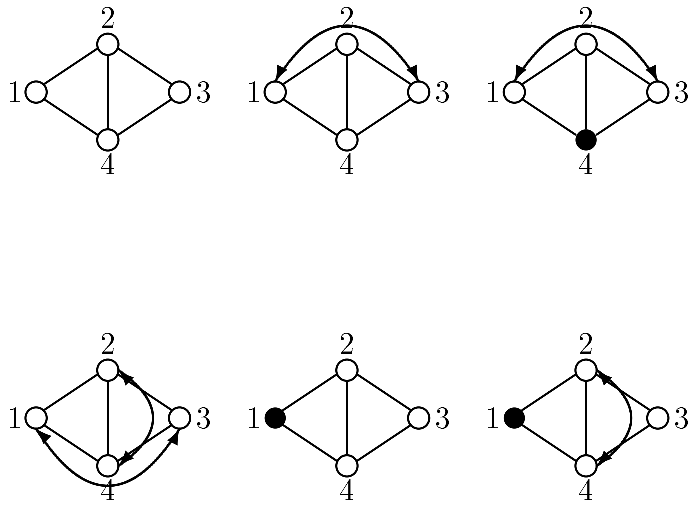
(a) $3 \sim 1$ *(by symmetry)* $\sim (1, 2, 4) \sim (3, 4)$

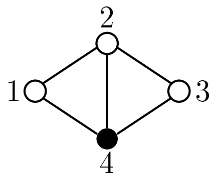
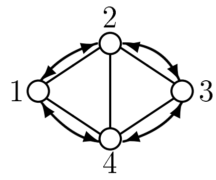
(b) $4 \sim 2$ *by symmetry*

Proof. By symmetry (diagram automorphism) and F_i algorithm, we get.

(a) $1 \sim (1, 2, 4)$ by $F\langle 1\rangle \sim (3, 4)$ by $F\langle 4\rangle$

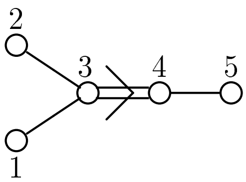
(b) $4 \sim (1, 3, 4)$ by $F\langle 4\rangle \sim (2, 3)$ by $F\langle 1\rangle$



$\square$

Example 2.3.9. The Dynkin diagram of a rank 5 hyperbolic Kac-Moody algebra is given by



Proposition 2.3.10. *The equivalenc classes of Vogan diagram of this rank 5 hyperbolic Kac-Moody algebra are given by*

(a) $1 \sim 2 \sim 3$

(b) $4 \sim (4, 5) \sim 5$

Proof. By using F_i algorithm

$$\begin{aligned} 2 &\sim (2,3) && \text{by } F\langle 2\rangle \\ &\sim (1,2,3,4) && \text{by } F\langle 3\rangle \end{aligned}$$

and

$$\begin{aligned} 1 &\sim (1,3) && \text{by } F\langle 1\rangle \\ &\sim (1,2,3,4) && \text{by } F\langle 3\rangle \\ &\sim (1,2,3,4,5) && \text{by } F\langle 4\rangle \end{aligned}$$

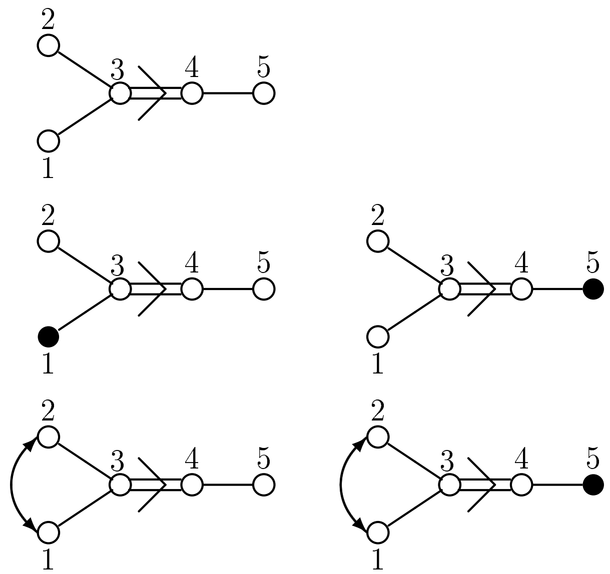
from above we get $1 \sim 2$

$3 \sim (1,2,3,4)$ by $F\langle 3\rangle \sim (1,2,4)$ by $F\langle 1\rangle$

So $1 \sim 2 \sim 3$

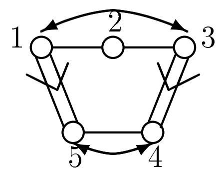
$4 \sim (4,5)$ by $F\langle 4\rangle \sim 5$ by F_5

So the Vogan Diagrams of the above Dynkin diagram with diagram automorphism become



□

Example 2.3.11. The Dynkin diagram of another rank 5 hyperbolic Kac-Moody algebra is given by



Proposition 2.3.12. *The equivalence classes of this rank 5 hyperbolic Kac-Moody algebra are given by*

(a) $2 \sim (1,2,3)$

(b) $5 \sim (4,5) \sim (2,3,4,5) \sim 4 \sim (4,5)$

(c) $1 \sim (1,2,5) \sim (2,3,5) \sim 3 \sim (2,3,4) \sim (1,2,4)$

Proof. (a) $(2) \sim (1,2,3)$ by $F\,(2) \sim (1,4,5)$

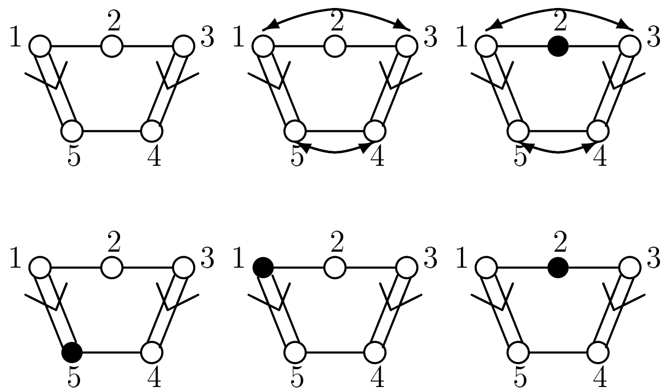
(b) $(5) \sim (4,5)$ by $F\,(5) \sim (2,3,4,5)$ by $F\,(3)$

and $(4) \sim (4,5)$ by $F\langle 4\rangle$

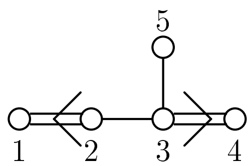
(c) $(1) \sim (1,2,5)$ by $F\langle 1\rangle \sim (2,3,5)$ by $F\langle 2\rangle \sim (2,3,4)$ by $F\langle 4\rangle$

and $3 \sim (2,3,4)$ by $F\langle 3\rangle \sim (1,2,4)$ by $F\langle 2\rangle$

□



Example 2.3.13. The Dynkin diagram of another rank 5 hyperbolic Kac-Moody algebra is given by



Proposition 2.3.14. *The equivalence classes of this rank 5 hyperbolic Kac-Moody algebra are given by*

(a) 1

(b) 4

(c) 5

(d) 3

(e) 2

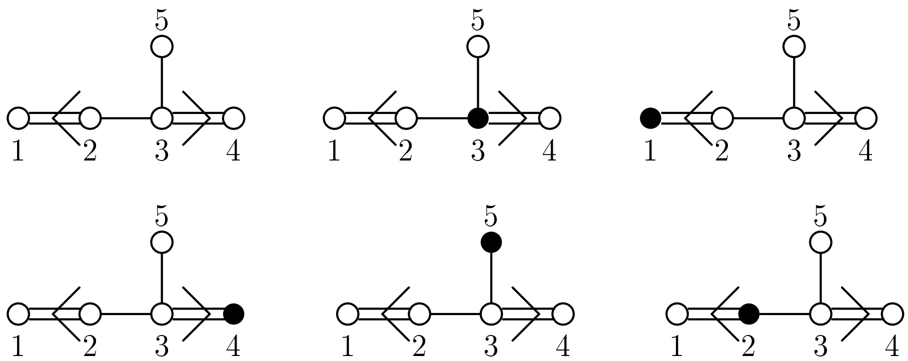
Proof. (a) $1 \sim 1$ by $F\langle 1 \rangle$

(b) $4 \sim 4$ by $F\langle 4 \rangle$

(c) $5 \sim (5, 3)$ by $F\langle 5 \rangle \sim (2, 3, 4)$ by $F\langle 3 \rangle$

(d) $3 \sim (2, 3, 4, 5)$ by $F\langle 5 \rangle$

(e) $2 \sim (1, 2, 3)$ by $F\langle 2 \rangle$



$\square$

The Vogan diagram of $HF_4^{(1)}$

Proposition 2.3.15. *The equivalence classes of this rank 6 hyperbolic Kac-Moody algebra are given by*

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

Proof. (a) $1 \sim (1,2) \sim (2,3) \sim (3,4) \sim (4,5) \sim (4,5,6) \sim (4,6)$ by $F\langle 1,2,3,4,5,6\rangle$

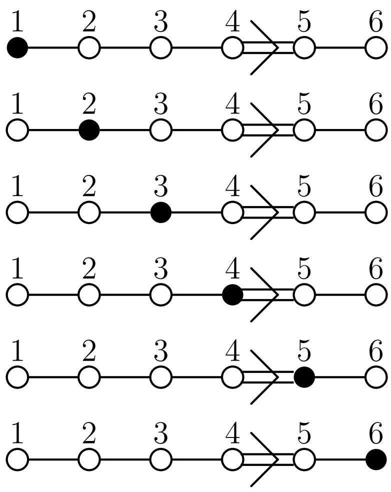
(b) $2 \sim (1,2,3) \sim (1,3,4) \sim (1,4,5) \sim (1,4,5,6) \sim (1,4,6)$ by $F\langle 2,3,4,5,6\rangle$

(c) $3 \sim (2,3,4) \sim (2,4,5) \sim (2,4,5,6) \sim (2,4,5,6) \sim (2,4,6)$ by $F\langle 3,4,5,6\rangle$

(d) $4 \sim (3,4,5) \sim (3,4,5,6) \sim (3,4,6)$ by $F\langle 4,5,6\rangle$

(e) $5 \sim (5,6) \sim 6$ by $F\langle 5,6\rangle$

□



Chapter 3

Vogan superdiagrams of basic Lie superalgebras

In this chapter we classify the real forms of Lie Superalgebra by Vogan superdiagrams, developing Borel de Seibenthal theorem of semisimple Lie algebras for Lie superalgebras. A Vogan superdiagram is a Dynkin diagram of triplet $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\bar{0}}, \triangle^{+})$, where $\mathfrak{g}_{\mathbb{R}}$ is a real Lie superalgebra, $\mathfrak{h}_{\bar{0}}$ cartan subalgebra, $\triangle^{+}$ positive root system. Although the classification of real forms of contragradient Lie superalgebras is already studied. Our method is a quicker and simpler one to classify.

3.1 Real forms of Basic Lie superalgebras

Definition 3.1.1. A finite dimensional Lie Superalgebra $\mathfrak{g}$ is called a basic classical Lie superalgebra if $\mathfrak{g}$ is a simple Lie superalgebra i.e, has no nontrivial $\mathbb{Z}_2$ graded ideal, $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and there exists a non-degenerate even supersymmetric invariant bilinear from on $\mathfrak{g}$

Proposition 3.1.2 ([46] Proposition 1.4)**.** *Let $\mathfrak{g}$ be a complex classical Lie superalgebra and let θ be an involutive semimorphism of $\mathfrak{g}$. Then* $\mathfrak{g}_{\mathbb{R}} = \{x + \theta x | x \in \mathfrak{g}\}$

is a real classical Lie superalgebra.

Proposition 3.1.3 ([46] Proposition 1.5)**.** *If $\mathfrak{g}_{\mathbb{R}}$ is a real classical Lie superalgebra, its complexification $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ is a Lie superalgebra which is either classical or direct sum of two isomorphic ideals which are classical*

Theorem 3.1.4 ([46] Theorem 4)**.** *Up to isomorphism, the real forms of the classical Lie superalgebras are uniquely determined by the real form $\mathfrak{g}_{\bar{0}\mathbb{R}}$ of the Lie subalgebra $\mathfrak{g}_{\bar{0}}$.*

The real form is said to standard (graded) when the real structure is standard (graded). Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of $\mathfrak{g}$ and let ω be the corresponding complex conjugation. Then $\omega|_{\mathfrak{g}_{\bar{0}}}$ is an antilinear involution of the Lie algebra $\mathfrak{g}_{\bar{0}}$. Hence there is a corresponding Cartan decomposition $\mathfrak{g}_{\bar{0}} = \mathfrak{t}_{\bar{0}} \oplus \mathfrak{p}_{\bar{0}}$, with Cartan involution θ.

Let $\mathring{\mathfrak{g}}$ be a semisimple Lie algebra and let ρ be a representation of $\mathring{\mathfrak{g}}$ in a vector space V. If $\mathring{\mathfrak{g}}_C$ is a real form of $\mathring{\mathfrak{g}}$ defined by the involutive semimorphism C, the representation ρ is said to be real for $\mathring{\mathfrak{g}}_C$ if there exists a semilinear transformations C_1 of V such that

$$\rho(Ca)C_1 = C_1\rho(a), \tag{3.1.1}$$

for all $a \in \mathring{\mathfrak{g}}$ and $C_1^2 = I$. If there exist C_1 satisfying Eq. (3.1.1) and such that $C_1^2 = -I$, then ρ is said to be antireal. If any C_1 satisfying Eq. (3.1.1) is singular, ρ is said to be areal. Both the natural representation π_l of A_1 and its contragradient π_l are real for the real form $\mathfrak{sl}(l+1, \mathbb{R})$, they are antireal for $\mathfrak{su}^*(l+1)$ and areal form $\mathfrak{su}(p.l+1-p)$. For $\mathfrak{sl}(m, n)$, the Lie subalgebra $\mathfrak{g}_\circ$ is the direct sum of its center K_0 and of the two simple ideals K_1 and K_2 of respectively type A_{m-1} and A_{n-1}. Hence, the only real forms of $K_1 + K_2$ for which the irreducible representation ρ' and ρ“ are real are $\mathfrak{sl}(m, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{su}^*(m) \oplus \mathfrak{su}^*(n)$ if m and n are even.

Similarly all other real form of this nature can be found in [46]

The following table gives a list of all the real forms associated with basic classical Lie superalgebras.

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{g}_{\bar{0}\mathbb{R}}$
$sl(m,n)$	$sl(m)\oplus sl(n)\oplus U(1)$	$sl(m\|n;\mathbb{R})$ $sl(m\|n;\mathbb{H})$ $su(p,m-p\|q,n-q)$	$sl(m,\mathbb{R})\oplus sl(n,\mathbb{R})\oplus\mathbb{R}$ $su^*(m)\oplus su^*(n)\oplus\mathbb{R}$ $su(p,m-p)\oplus su(q,n-q)\oplus i\mathbb{R}$
$sl(n,n)$	$sl(n)\oplus sl(n)$	$psl(n\|n;\mathbb{R})$ $psl(n\|n;\mathbb{H})$ $su(p.n-p\|q,n-q)$	$sl(n,\mathbb{R})\oplus sl(n,\mathbb{R})$ $su^*(n)\oplus su^*(n)$ $su(p,n-p)\oplus su(q,n-q)$
$\mathfrak{osp}(2m+1,2n)$ $\mathfrak{osp}(0,2n)$	$so(2m+1)\oplus sp(2n)$ $sp(2n)$	$osp(p,2m+1-p\|2n;\mathbb{R})$ $osp(1\|2n;\mathbb{R})$	$so(p,2m+1-p)\oplus sp(2n;\mathbb{R})$ $sp(2n,\mathbb{R})$
$\mathfrak{osp}(2,2n)$	$so(2)\oplus sp(2n)$	$osp(p\|2n;\mathbb{R})$ $osp(2\|2q,2n-2q;\mathbb{H})$	$so^*(2)\oplus sp(2n;\mathbb{R})$ $so^*(2)\oplus sp(2q,2n-2q)$
$\mathfrak{osp}(2m,2n)$	$so(2m)\oplus sp(2n)$	$osp(p,2m-p\|2n;\mathbb{R})$ $osp(2m\|2q,2n-2q;\mathbb{H})$	$so(p,2m-p)\oplus sp(2n;\mathbb{R})$ $so^*(2m)\oplus sp(2q,2n-2q)$ $sp(n)\oplus so^*(2m)$ $so(2m)\oplus so^*(2n,\mathbb{R})$
$F(4)$	$sl(2)\oplus so(7)$	$F(4;0)$ $F(4;3)$ $F(4;2)$ $F(4;1)$	$sl(2,\mathbb{R})\oplus so(7)$ $sl(2,\mathbb{R})\oplus so(1,6)$ $sl(2,\mathbb{R})\oplus so(2,5)$ $sl(2,\mathbb{R})\oplus so(3,4)$
$G(3)$	$sl(2)\oplus G_2$	$G(3,0)$ $G(3,1)$	$sl(2,\mathbb{R})\oplus G_{2,0}$ $sl(2,\mathbb{R})\oplus G_{2,2}$
$D(2,1;\alpha)$	$sl(2)\oplus sl(2)\oplus sl(2)$	$D(2,1;\alpha;0)$ $D(2,1;\alpha;1)$ $D(2,1;\alpha;2)$	$sl(2,\mathbb{R})\oplus sl(2,\mathbb{R})\oplus sl(2,\mathbb{R})$ $sl(u)\oplus sl(u)\oplus sl(u)$ $sl(2,\mathbb{C})\oplus sl(2,\mathbb{R})$

Table - 6

3.2 Vogan diagrams of Basic Lie Superalgebras

One can know the from Dynkin diagram that the only diagram automophism possibilies are in $\mathfrak{sl}(m,n)$, $\mathfrak{osp}(2m,2n)$, $D(2,1;\alpha)$, so we use the following lemma for the existence.

Lemma 3.2.1 (Lemma 4.7 [34]). *There exists $\sigma \in Aut(\mathfrak{g})$ such that $\sigma|_{g_0}$ is a nontrivial automorphism if and only if $\mathfrak{g}$ is of type $\mathfrak{sl}(m,n)$, $\mathfrak{osp}(2m,2n)$, $D(2,1;\alpha)$,$\alpha \in \{1,(-2)^{\pm 1}\}$ and $\sigma|_{g_0}$is as follows.*

1. *If $\mathfrak{g}$ is of type $\mathfrak{sl}(m,n)$ with $n \neq m$, then $\sigma|_{\mathfrak{g}_0}$ restricts to the nontrivial diagram automorphism of both A_n and A_m.*

2. *If $\mathfrak{g}$ is of type $\mathfrak{sl}(n,n)$, then $\sigma|_{\mathfrak{g}_0}$ is either the nontrivial diagram automorphism of both A_n components, or it is the flip automorphism between the two A_n components, or the composition of these two automorphisms.*

3. *If $\mathfrak{g}$ is of type $\mathfrak{osp}(2m,2n)$, $m>2$, then $\sigma|_{\mathfrak{g}_0}$ is the unique diagram automorphism of $\mathfrak{g}_0$.*

4. *If $\mathfrak{g}$ is of type $D(2,1) \cong D(2,1,a)$, $a \in 1,(-2)^{\pm 1}$, then θ is the unique diagram automorphism of the diagram.*

We frame the following concepts.

Definition 3.2.2. An abstract Vogan diagram is an abstract Dynkin diagram with two pieces of additional structure, one is an automorphism of order 1 or 2 of the diagram, which is to be indicated by labeling the 2 element orbits. The other is the subset of the 1 element orbits which is to be indicated by painting the vertices corresponding to the members of the subset of noncompact roots. Every Vogan diagram is of course an abstract Vogan diagram of Lie superalgebra.

Definition 3.2.3. The Vogan superdiagram of Lie superalgebras is the Vogan diagram of the even part of Lie superalgerbas. In addition to that

(a) The vertices fixed by the Cartan involution of the even part is painted (or unpainted) depending whether the the root is noncompact (or compact).

(b) Label the 2-elements orbit by the diagram automorphism indicated with two sided arrow.

(c) The odd root remain unchanged.

Theorem 3.2.4. *If an abstract Vogan diagram is given, then there exist a real Lie superalgebra* $\mathfrak{g}_{\mathbb{R}}$*, a Cartan involution* θ*, a maximally compact* θ *stable Cartan subalgebra and a positive system* $\triangle_{\bar{0}}^{+}$ *for* $\triangle = \triangle(\mathfrak{g}, \mathfrak{h})$ *that takes* $it_{\bar{0}}$ *before* $a_{\bar{0}}$ *such that the given diagram is the Vogan diagram of* $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\bar{0}}, \triangle_{0}^{+})$.

Remark 3.2.5. Briefly the theorem says that any abstract Vogan diagram comes from some $\mathfrak{g}_{\bar{0}}$. Thus the theorem is an analog for real semisimple Lie algebras of the existence theorem for complex semisimple Lie algebras.

We will modify the Borel and de Siebenthal Theorem for Lie superalgebra.

Theorem 3.2.6. *Let* $\mathfrak{g}_{\mathbb{R}}$ *be a non complex real Lie superalgebra and the Vogan diagram of* $\mathfrak{g}_{\mathbb{R}}$ *be given that correponding to the triple* $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_0, \triangle^{+})$*. Then* $\exists$ *a simple system* $\prod'$ *for* $\triangle = \triangle(\mathfrak{g}, \mathfrak{h})$*, with corresponding positive system* $\triangle^{+}$*, such that* $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\bar{0}}, \triangle^{+})$ *is a triple and there is at most single painted simple roots for each even part in its Vogan diagram. Furthermore suppose the automorphism associated with the Vogan diagram is the identity, that* $\prod' = \alpha_1, \cdots, \alpha_l$ *and that* $\omega_1, \cdots, \omega_l$ *is the dual basis for each even part such that* $\langle \omega_j, \alpha_k \rangle = \delta_{jk}/\epsilon_{kk}$*, where* ϵ_{kk} *is the diagonal entries to make cartan matrix symmetric. The the single painted simple root for each even parts may be chosen so that there is no* i' *with* $\langle \omega_i - \omega_{i'}, \omega_{i'} \rangle > 0$.

Proof. We know $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. The positive even root system $\triangle_{0}^{+}$ can be written as

$$\triangle_{0}^{+} = \triangle_{01}^{+} \cup \triangle_{02}^{+}$$

where $\triangle_{01}^{+}$ are the even positive root system for simple root system formed by e_i basis and $\triangle_{02}^{+}$ are for δ_j basis. For the even part, we take $< \omega_i, \alpha_j >= \delta_{ij}/\epsilon_{kk}$.

This makes the Cartan matrix symmetric and so that we can get the inverse of Cartan matrix of A_m and A_n for $\mathfrak{sl}(m, n)$ Lie superalgebra. Similarly construction will follow for other Lie superalgebras. Each inverse for even part is associated with the dual basis ω. For the odd part the condition is $<\omega_i, \alpha_j>=\delta_{ij}$ and we donot get any painted vertices. The symmetrizable condition of Kac-Cartan matrix gives $S = \epsilon_{kk}A$ where S is the symmetric Cartan matrix. The table given below the values of ϵ_{kk} for different superalgebras.

Lie superalgebra	ϵ_{kk}
$\mathfrak{sl}(m, n)$	$(1, \cdots, 1, -1, \cdots, -1)$
$\mathfrak{osp}(2m+1, 2n)$	$(1, \cdots, 1, -1 \cdots, -1, -2)$
$\mathfrak{osp}(0, 2n)$	$1, \cdots, 1, 2$
$C(n)$	$(-1, 1, \cdots, 1, \frac{1}{2})$
$\mathfrak{osp}(2m, 2n)$	$(1, \cdots, 1, -1-1, \cdots, -1, -1, -1)$
$D(2, 1; \alpha)$	$(1, -1, \frac{1}{a})$
$F(4)$	$(-1, 1, \frac{1}{2})$
$G(3)$	$(-\frac{1}{2}, 1, \frac{1}{3},)$

Taking suitable normalization condition for each type of Lie superalgebras and from the two Lemmas 6.97 and 6.98 [36] we get redundancy test for each even part. So now the Vogan diagram of Lie superalgebras becomes single painted (for each even part) vertices Vogan diagram. □

Lemma 3.2.7 (Lemma 6.97 [36] modified)**.** *Let $\triangle$ be an irreducible abstract reduced root system in a real vector superspace V, let Π_{01} and Π_{02} be the two simple simple root system for even parts e_i and δ_j bases respectivily and let ω and ω' be nonzero members of V that are domiant relative to Π_i's. Then $\langle \omega, \omega' \rangle > 0$.*

Proof. Using the suitable normalizations of e_i and δ_j we get the proof of the Lemma as follows. For Π_{01}, we have to show first $\omega = \sum_{\alpha \in \Pi_{01}} a_\alpha \alpha$, all the a_α are

≥ 0 and for Π_{02}; $\omega = \sum_{\alpha \in \Pi_{02}} b_\alpha \alpha$, all the b_α are ≥ 0. Let us enumerate Π_{01} as $\{\alpha_1, \cdots, \alpha_m\}$ so that

$$\omega = \sum_{i=1}^{r} a_i \alpha_i - \sum_{i=r+1}^{s} d_i \alpha_i = \omega^+ - \omega^-$$

Since $\omega^- = \omega^+ - \omega$, we have

$$0 \leq |\omega^-|^2 = \langle \omega^+, \omega^- \rangle - \langle \omega^-, \omega \rangle = \sum_{i=1}^{r} \sum_{i=r+1}^{s} a_i d_j \langle \alpha_i, \alpha_j \rangle - \sum_{i=r+1}^{s} d_j \langle \omega, \alpha_j \rangle.$$

The two terms of right side including the $-$ sign is term by term $\leq$ by hypothesis. Therefore we conclude that $\omega^- = 0$ for Π_{01}.

Similarly for Π_{02} i.e., δ_i basis; Let us enumerate Π_{02} as $\{\alpha_1, \cdots, \alpha_n\}$

$$0 \leq |\omega^-|^2 = \langle \omega^+, \omega^- \rangle - \langle \omega^-, \omega \rangle = \sum_{i=1}^{r} \sum_{i=r+1}^{s} b_i d_j \langle \alpha_i, \alpha_j \rangle - \sum_{i=r+1}^{s} d_j \langle \omega, \alpha_j \rangle.$$

and using $\langle \omega_j, \alpha_k \rangle = \delta_{jk}/\epsilon_{kk}$ we get the first term on the right side ≤ 0 using suitable normalization of δ_i. Again for at least one index i $\langle \alpha_i, \omega' \rangle > 0$ for each even part Lie superalgebras. Since $\omega' \neq 0$. Then

$$\langle \omega, \omega' \rangle = \sum_{i} a_j \langle \alpha_j, \omega' \rangle \geq a_i \langle \alpha_i, \omega' \rangle > 0$$

This proves the lemma. $\square$

Lemma 3.2.8 (Lemma 6.98 [36] modified)**.** *Let $\mathfrak{g}_{0\mathbb{R}}$ be a noncomplex simple real Lie superalgebra and let the Vogan diagram of $\mathfrak{g}_{0\mathbb{R}}$ be given that corresponding to the triple $(\mathfrak{g}_0, \mathfrak{h}_0, \triangle^+)$. Write $\mathfrak{h}_{01} = \mathfrak{t}_{01} \oplus \mathfrak{a}_{01}$ and $\mathfrak{h}_{02} = \mathfrak{t}_{02} \oplus \mathfrak{a}_{02}$ for two even parts. Let V be the span of simple roots that are imaginary, let $\triangle_0$ be the root system $\triangle \cap V$, let $\mathcal{H}$ be the subset of $i\mathfrak{t}_0$ paired with V and let Λ be the subset of $\mathcal{H}$ where* *all roots of $\triangle_0$ take integer values and all noncompact roots of $\triangle_0$ take odd integer values. Then Λ is nonempty. In fact if $\alpha_1, \cdots, \alpha_m$ is any simple system for $\triangle_0$ and if $\omega_1, \cdots, \omega_m$ in V are defined by $\langle \omega, \alpha_k \rangle = \delta_{jk}/\epsilon_{kk}$, then the element*

$$\omega = \sum_{i \text{ with } \alpha_i \text{ noncompact}} \omega_i$$

Proof. Fix a simple system $\alpha_1, \cdots, \alpha_m$ for $\Delta_{\bar{0}}$ and let $\Delta_{\bar{0}}^{+}$ be the set of positive roots of $\Delta_{\bar{0}}$. Define $\omega_1, \cdots, \omega_m$ by $\langle \omega, \alpha_k \rangle = \delta_{jk}/\epsilon_{kk}$. If $\alpha = \sum_{i=1}^{m} n_i \alpha_i$ is a positive root of $\Delta_{\bar{0}}$, then $\langle \omega, \alpha \rangle$ is the sum of the n_i for which α_i is noncompact.

Using induction of the Lemma 6.98 [36] for even part of root system the above Lemma will be proved and each even roots satisfy

compact root $+$ compact root $=$ compact root

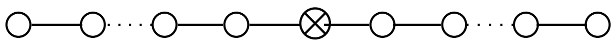
compact root$+$ noncompact root $=$ noncompact root

noncompact root $+$ noncompact root $=$ noncompact root

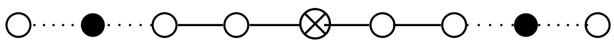
$\square$

1. $\mathfrak{sl}(m, n)$:

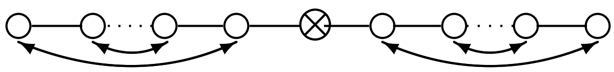
   The Vogan diagrams and real foms of Lie superalgebras $\mathfrak{sl}(m, n)$ are as follows.



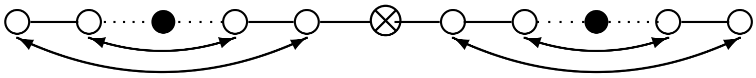
$\mathfrak{sl}(n, \mathbb{C})$



$\mathfrak{su}(p, m-p)$ $\mathfrak{su}(r, n-r)$



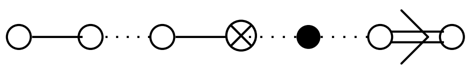
$\mathfrak{sl}(m, \mathbb{R})$ $\mathfrak{sl}(n, \mathbb{R})$



$\mathfrak{su}^*(m)$ $\mathfrak{su}^*(n)$

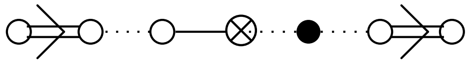
2. $\mathfrak{osp}(2m+1, 2n)$:

   The Lie subalgebra of $\mathfrak{g}_0$ is $C_m \oplus B_n$ The only trivial automophism of even part of Vogan diagram of $\mathfrak{osp}(2m+1, 2n)$ is shown below and the real form is $\mathfrak{sp}(2n, \mathbb{R}) \oplus \mathfrak{so}(p, 2m+1-p)$



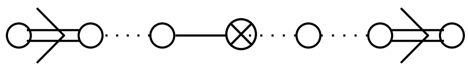
$so(p, 2m+1-p)$

   Because of missing of real form of the first even part, we need and additional C_n Dynkin diagram, so the Vogan diagrams become



$sp(2n, \mathbb{R})$ $so(p, 2m+1-p)$

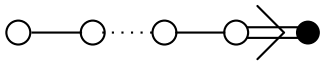
   The another real form of Lie superalgebra $\mathfrak{osp}(2m+1, 2n)$ is $sp(2n, \mathbb{R}) \oplus so(2m+1)$ and the corresponding Vogan diagram is drawn below.



$sp(2n, \mathbb{R})$ $so(2m+1)$

3. $\mathfrak{osp}(0, 2n)$:

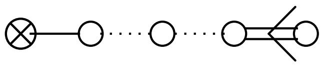
   The Vogan diagram below is a unpainted diagram but it consists of its own painted vertices on the extreme right.



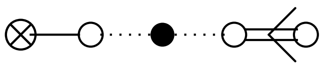
$sp(2n, \mathbb{R})$

4. $\mathfrak{osp}(2, 2n-2)$:

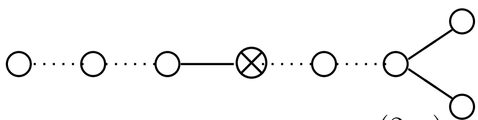
   The unpainted Vogan diagram of $C(n+1)$ designates the real form $so^*(2) \oplus sp(2n, \mathbb{R})$.



   From the trivial automophism of the even part of $C(n+1)$ we can draw the Vogan diagram and the real form $so^*(2) \oplus sp(2q, 2n-2q)$.

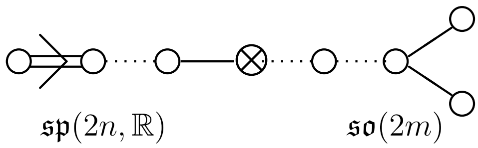


5. $\mathfrak{osp}(2m, 2n)$: The Lie subalgebra of $\mathfrak{g}_0$ is $C_m \oplus D_n$. The compact real form of C_m is $\mathfrak{sp}(m)$. The real form $sp(2n, \mathbb{R}) \oplus so(2m)$ of the abstract Vogan diagram of $\mathfrak{osp}(2m, 2n)$ for the above subalgebras is followed below.

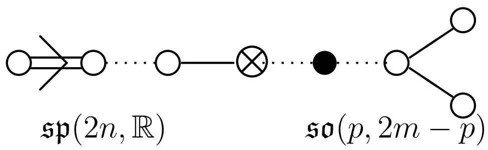


$\mathfrak{so}(2m)$

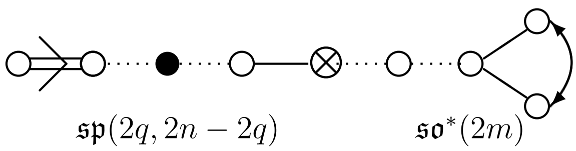
   Since from the diagram we get only $\mathfrak{so}(2m)$ part of real form, so for the $\mathfrak{sp}(2n)$ part we need the addition C_n diagram in the diagram above. Subsequently the Vogan diagram becomes



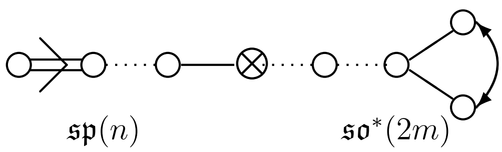
The first trivial involution for one of the even part for the Vogan diagram of $\mathfrak{osp}(2m, 2n)$ is given below and the real form of this diagram is $\mathfrak{sp}(2n, \mathbb{R}) \oplus \mathfrak{so}(p, 2m-p)$.



The nontrivial involution for the Vogan diagram of $\mathfrak{osp}(2m, 2n)$ is given below and the real form of this diagram is $sp(2q, 2n-2q) \oplus so^*(2m)$.

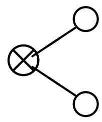


The below Vogan diagram is formed by the nontrivial involution . The real form for this superlagebra is $\mathfrak{sp}(n) \oplus \mathfrak{so}^*(2m)$.



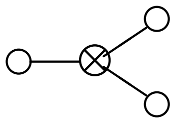
6. $D(2,1;\alpha)$:

The unpainted and no two element orbit Vogan diagram is given below with the real form. Since we can get only the real form $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ from ordinary Vogan diagram

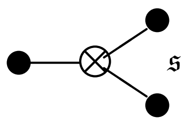


$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

So our requisite Vogan diagrams for the suitable real forms are

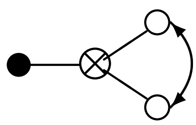


$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$



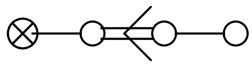
$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$

The nontrivial involution of the Dynkin diagram of $D(2,1;\alpha)$ makes the following Vogan diagram as shown below.

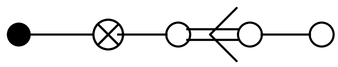


$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{R})$

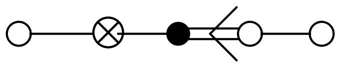
7. $F(4)$: From the Dynkin diagram we can get only the real form $so(7)$ and the Vogan diagram



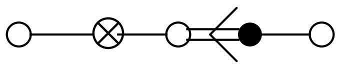
we add the extra even part root to get the desired real forms and Vogan diagrams.



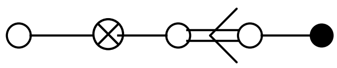
$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(7)$



$\mathfrak{su}(2) \oplus \mathfrak{so}(1, 6)$



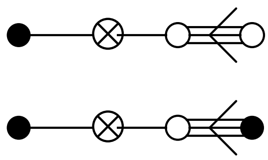
$\mathfrak{su}(2) \oplus \mathfrak{so}(2, 5)$



$\mathfrak{su}(2) \oplus \mathfrak{so}(3, 4)$

8. Case $G(3)$:

   The Vogan diagram of $G(3)$ with real form $\mathfrak{sl}(2, \mathbb{R}) \oplus G_{2,0}$ and $\mathfrak{sl}(2, \mathbb{R}) \oplus G_{2,2}$ are



Chapter 4

Vogan diagrams of untwisted affine Lie superalgebras

In this Chapter we classify the Vogan diagrams of untwisted affine Kac-Moody superalgebras superalgebras.

4.1 A Realization of Affine Kac-Moody superalgebras

Let $L = \mathbb{C}[t, t^{-1}]$ be an algebra of Laurent polynomials in t . The residue of a Laurent polynomial $P = \sum_{k \in \mathbb{Z}} c_k t^k$ (where all but a finite number of c_k are 0) is defined as $ResP = c_{-1}$. Let $\mathcal{G}$ be a finite dimensional simple Lie superalgebra ($\mathcal{G} \neq gl(n|n),$), $(.|.)$ be a nondegenerate invariant symmetric bilinear form on $\mathcal{G}$. The definition of affine untwisted B.S.A. $\mathcal{G}^{(1)}$ follows from affine algebras, i.e. $\mathcal{G}^{(1)}$ is the loop algebra constructed from $\mathcal{G}$. Define an infinite dimensional superalgebra $\mathcal{G}^{(1)}$ as $\mathcal{G} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}D \oplus \mathbb{C}K$; and bracket is defined by

$$[X \otimes t^k, Y \otimes t^l] = [X, Y] \otimes t^{k+l} + k\delta_{k,-l}(X, Y)K,$$

$$[D, K] = 0$$

,

$$[D, X \otimes t^k] = kX \otimes t^k$$

A simple root system of an affine B.S.A. $\mathcal{G}^{(1)}$ is obtained from a simple root system $\triangle_s$ of $\mathcal{G}$ by adding to it the affine root which project on $\triangle_s$ as the corresponding lowest root. The simple root systems of $\mathcal{G}^{(1)}$ are therefore associated to the extended Dynkin diagrams used to determine the regular subsuperalgebras.

4.2 Cartan Involution and Invariant bilinear form

An involution θ of a real semisimple Lie algebra $\mathfrak{g}_0$ such that symmetric bilinear form

$$B_\theta(X, Y) = -B(X, \theta Y)$$

is positive definite is called a *Cartan involution*.

For Contragradient Lie superalgebras there exist a supersymmetric nondegenerate invariant bilinear form on it and defined in [13] as

$$B_\theta(X, Y) = B(\theta X, \theta Y)$$

Let $\mathcal{G}^{(1)}$ be a complex affine untwised Kac-Moody superalgebra. The uniqueness of B is extended to $\mathcal{G}^{(1)}$. The killing form is unique when restricted to $\mathcal{G}_0$.

An involution θ for affine untwisted Kac-Moody superalgebras is defined by

$$\theta(t^m \otimes X) = t^m \otimes \theta(X)$$

We say a real form has Cartan automorphism θ if B restricts to the Killing form on $t^m \otimes X$ where $X \in \mathcal{G}_0$ and B_θ is symmetric negative definite on $\mathcal{G}^1$. A bilinear supersymmetric invariant form $B^1(.,.)$ can be set up on $\mathcal{G}^{(1)}$ by the definitions

$$B^{(1)}(t^j \otimes X, t^k \otimes Y) = \delta^{j+k} B(X,Y)$$

$$B^{(1)}(t^j \otimes X, K) = 0$$

$$B^{(1)}(t^j \otimes X, D) = 0$$

$$B^{(1)}(K,K) = 0$$

$$B^{(1)}(K,D) = 1$$

$$B^{(1)}(D,D) = 1$$

We say that a real form of $\mathcal{G}$ has Cartan automorphism $\theta \in \mathrm{aut}_{2,4}(\mathcal{G})$ (automophism of order 2 on the even part and automophism of order 4 on the odd part) if $B^{(1)}$ restricts to the Killing form on $\mathcal{G}_0$ and $B_\theta^{(1)}$ is symmetric negative definite on $\mathcal{G}_{\mathbb{R}}$.

Proposition 4.2.1. *Let $\theta \in aut_{2,4}(\mathcal{G}^{(1)})$. Then there exists a real form $\mathcal{G}_{\mathbb{R}}^{(1)}$ such that θ restricts to a Cartan automorphism on $\mathcal{G}_{\mathbb{R}}^{(1)}$.*

Proof. There exist θ such that

$$\theta(K) = K$$

and

$$\theta(D) = D$$

Since θ is an $\mathcal{G}^{(1)}$ automorphism, it preserves B, namely

$$B^{(1)}(X,Y) = B^{(1)}(\theta X, \theta Y)$$

. $B_\theta^{(1)}(X,Y) = B_\theta^{(1)}(Y,X),\ B_\theta^{(1)}(X,Y) = B_\theta^{(1)}(\theta X, \theta Y),\ B_\theta^{(1)}(X,\theta X) = 0$

$$B_\theta^{(1)}(X\otimes t^m, Y\otimes t^n) = B_\theta^{(1)}(Y\otimes t^n, X\otimes t^m) =$$

$$B^{(1)}(X\otimes t^m, Y\otimes t^n) = t^{m+n}B^{(1)}(X,Y)$$

for all $X, Y \in \mathcal{G}_0$.

$$B^{(1)}(K, X\otimes t^k) = B^{(1)}(D, X\otimes t^k) = B^{(1)}(D,D) = B^{(1)}(K,K) = 0$$

. For $Z \in L(t,t^{-1})\otimes\mathcal{G}_0$ and $X, Y \in L(t,t^{-1})\otimes\mathcal{G}_1$

$$B_\theta^{(1)}(X,[Z,Y]) = B^{(1)}(X,[\theta Z,\theta Y]) = -B_\theta^{(1)}(X,[\theta Z,\theta Y])$$

and

$$B_\theta^{(1)}(X,[Z,Y]) = 0$$

$\forall\ X \in \mathbb{C}K$ or $\mathbb{C}D$

$\mathcal{G}_{\mathbb{R}}^{(1)} \simeq \mathcal{G}_{\bar{0}\mathbb{R}}^{(1)} \simeq \mathcal{G}_{\bar{0}\mathbb{R}}$. The above three real forms are isomorphic. So the Cartan decomposition of $\mathcal{G}_{\mathbb{R}}^{(1)}$ are isomorphic to $\mathcal{G}_{\bar{0}}$ and $\mathcal{G}_{\bar{0}} = \mathfrak{k}_0 \oplus \mathfrak{p}_0$.

$$B_\theta^{(1)}(X,[Z,Y]) = \begin{cases} -B_\theta^{(1)}([Z,X],Y) & \text{if } Z \in \mathfrak{k}_0 \\ B_\theta^{(1)}([Z,X],Y) & \text{if } Z \in \mathfrak{p}_0 \end{cases}$$

It is known that a real form of $\mathcal{G}$ has Cartan automorphism $\theta \in$aut$_{2,4}(\mathcal{G})$ if $B^{(1)}$ restricts to the Killing form on $\mathcal{G}_0$ and $B_\theta^{(1)}$ is symmetric negative definite on $\mathcal{G}_{\mathbb{R}}$. $B_\theta^{(1)}(X_i,X_j) = \delta_{ij}$. It follows that $B_\theta^{(1)}$ negative definite on $L(t,t^{-1})\otimes\mathcal{G}^{(1)}$. But $B_\theta^{(1)}$ is symmetric bilinear form on $L_1 \supset \mathbb{R}\{1\otimes iX_1, 1\otimes iX_2, \cdots, d\}$. So it is concluded that θ is a Cartan automorphism on $\mathcal{G}^{(1)}$. $\square$

4.3 Vogan diagram

The uniqueness of Cartan automorphism θ from Dynkin diagram of $\mathcal{G}_0$ to $\mathcal{G}^{(1)}$ proved in [13]. This gives a straightforward proof of the above theory to affine untwisted Kac-Moody superalgebras cases.

Definition 4.3.1. A *Vogan diagram* (p, d) on Dynkin diagram D (where p is the painting (black or white) on the diagram and d is the diagram involution) of $\mathcal{G}^{(1)}$ and one of the following holds:

(i) θ fixes grey vertices.

(ii) θ interchanges grey vertices and $\sum_S a_\alpha$ is odd (S is the d orbit vertices).

(iii) $\sum_S a_\alpha$ is odd.

Let inv$(\cdot)$ denote the involution on, and aut$_{2,4}(\cdot)$ denote automophisms on a Kac-Moody superalgebra of order 2 on the even part and order 4 on the odd part.

Proposition 4.3.2. *Let $\mathcal{G}_\mathbb{R}$ be a real form, with Cartan involution $\theta \in inv(\mathcal{G}_\mathbb{R})$ and Vogan diagram (p, d) of D_0 (set of even vertices). The following are equivalent:*

(i) θ extends to $aut_{2,4}\mathcal{G}^{(1)}$.

(ii) $(\mathcal{G}_{\bar{0}\mathbb{R}})$ extends to a real form of $\mathcal{G}^{(1)}$.

(iii) (p, d) extends to a Vogan diagram on D.

Proof. Let S be the d- orbits of vertices defined by [9] S ={vertices painted by p}∪{white and adjacent 2-element d-orbits}∪{grey and non adjacent 2-element d-orbits} Let D be the Dynkin diagram of $\mathcal{G}^{(1)}$ of simple root system $\Phi \cup \phi$ (Φ simple root system with ϕ lowest root) with $D = D_{\bar{0}} + D_{\bar{1}}$, where $D_{\bar{0}}$ and $D_{\bar{1}}$ are respectively the white and grey vertices. The numerical label of the diagram shows that $\sum_{\alpha\in D_{\bar{1}}} = 2$ has either two grey vertices with label 1 or one grey vertex with label 2. For $\{\gamma, \delta\}$ odd vertices with the properties with the numerical labelling are

(i) $D_{\bar{1}} = \{\gamma, \delta\}$, so the labelling of the odd vertices are 1.

(ii) $D_{\bar{1}} = \{\gamma\}$, so labelling is 2 ($a_\alpha = 2$) on odd vertex.

$\theta \in$inv($\mathcal{G}_{\mathbb{R}}$); θ permutes the weightspaces $L(t, t^{-1}) \otimes \mathcal{G}_{\bar{1}}$ The rest part of proof of the proposition follows from the proof of the Propostion 2.2 of [13].. $\square$

4.4 Root systems of affine untwised Kac-Moody superalgebra ($\mathcal{G}^{(1)}$)

Let Π be the simple root system of $\mathcal{G}^{(1)}$ given by

$$\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_m, \alpha_{m+1}, \cdots \alpha_{m+n+1}, \alpha_0 = \delta - \theta\}.$$

(i) $\mathfrak{sl}(m+1, n+1)^{(1)}$

$\triangle = \{\alpha_0 = k + \delta_{n+1} - e_1, \alpha_1 = e_1 - e_2, \cdots, \alpha_m - e_{m+1}, \alpha_{m+1} = e_{m+1} - \delta_1, \alpha_{m+2} = \delta_1 - \delta_2, \cdots, \alpha_{n+m} = \delta_n - \delta_{n+1}\}.$

where k is the isotropic affine direction.

(ii) $\mathfrak{osp}(2m+1, 2n)^{(1)}(m > 2)$

$\triangle = \{k - 2\delta_1, \alpha_1 = \delta_1 - \delta_2, \alpha_2 = \delta_2 - \delta_3, \cdots, \alpha_n = \delta_n - e_1, \alpha_{n+1} = e_1 - e_2\}.$

(iii) $\mathfrak{osp}(2m, 2n)^{(1)}(m > 2)$

$\triangle = \{k - 2\delta_1, \alpha_1 = \delta_1 - \delta_2, \alpha_2 = \delta_2 - \delta_3, \cdots, \alpha_n = \delta_n - e_1, \alpha_{n+1} = e_1 - e_2, \alpha_{n+m-1} = e_{m-1} - e_m, \alpha_{n+m-1} = e_{m-1} + e_m\}.$

(iv) $C(n)^{(1)}$

$\triangle = \{\alpha_0 = k - e - \delta_1, \alpha_1 = e - \delta_1, \alpha_2 = \delta_1 - \delta_2, \cdots \alpha_n = \delta_{n-1} - \delta_n, \alpha_{n+1} = 2\delta_{n-1}\}$.

(v) $D(2,1,\alpha)^{(1)}$

$\triangle = \{\alpha_0 = k - (e_1 + e_2 + e_3), e_1 - e_2 - e_3, 2e_2, 2e_3\}$.

(vi) $F(4)^{(1)}$

$\triangle = \{\alpha_0 = k - 3\delta, \delta + \frac{1}{2}(-e_1 - e_2 - e_3), e_3, e_2 - e_3, e_1 - e_2)\}$.

(vii) $G(3)^{(1)}$

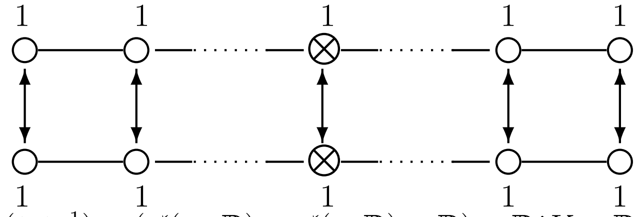
$\triangle = \{\alpha_0 = k - 4\delta, \delta + e_1, e_2, e_3 - e_2\}$

The Cartan subalgebra of $\mathcal{G}^{(1)}$ is

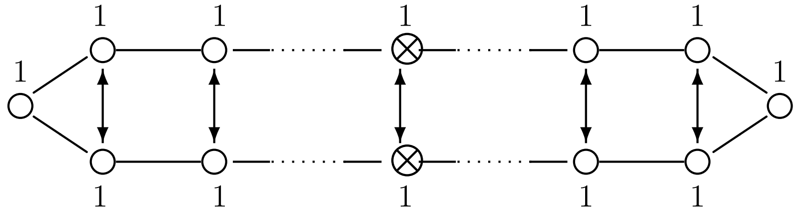
$$\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}D$$

4.5 Real forms from Vogan diagram of untwisted affine Lie superalgebras

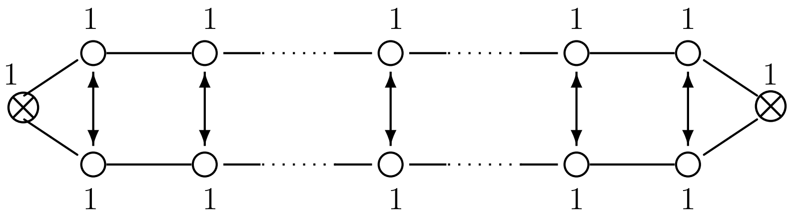
(i) $\mathfrak{sl}(m+1, n+1)^{(1)}$



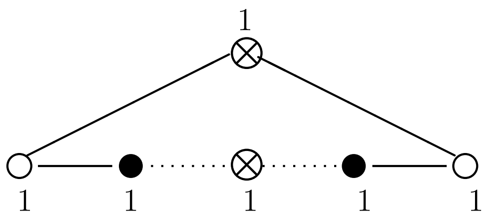
$L(t,t^{-1}) \otimes (\mathfrak{sl}(m,\mathbb{R}) \oplus \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R}) \oplus \mathbb{R}iK \oplus \mathbb{R}iD$



$L(t,t^{-1}) \otimes (\mathfrak{su}^*(m) \oplus \mathfrak{su}^*(n,\mathbb{R}) \oplus \mathbb{R}) \oplus \mathbb{R}iK \oplus \mathbb{R}iD$

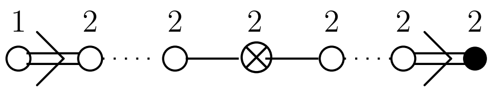


$L(t, t^{-1}) \otimes (\mathfrak{sl}(n, \mathbb{C})) \oplus \mathbb{R}) \oplus \mathbb{R}iK \oplus \mathbb{R}iD$

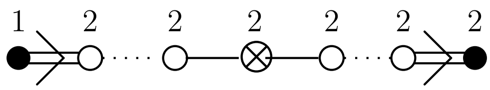


$L(t, t^{-1}) \otimes (\mathfrak{su}(p, m-p, \mathbb{R}) \oplus \mathfrak{su}(r, n-r, \mathbb{R}) \oplus i\mathbb{R}) \oplus \mathbb{R}ic \oplus \mathbb{R}id$

(ii) $\mathfrak{osp}(2m+1, 2n)^{(1)}$

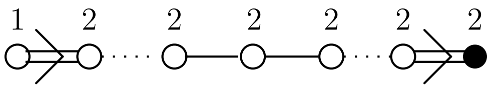


$L(t, t^{-1}) \otimes (\mathfrak{sp}(m, \mathbb{R})) \oplus \mathbb{R}ic \oplus \mathbb{R}id$

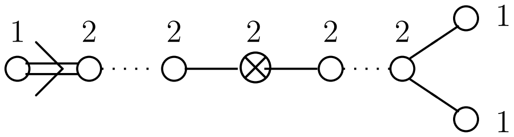


$L(t, t^{-1}) \otimes (\mathfrak{sp}(m, \mathbb{R}) \oplus \mathfrak{so}(p, q)) \oplus \mathbb{R}ic \oplus \mathbb{R}id$

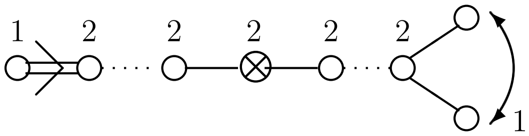
(iii) $\mathfrak{osp}(1, 2n)^{(1)}$

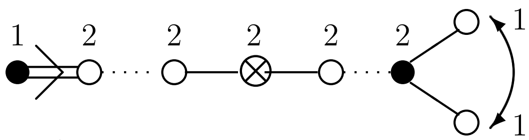


(iv) $\mathfrak{osp}(2m, 2n)^{(1)}$



$L(t, t^{-1}) \otimes (\mathfrak{sp}(r, s) \oplus \mathfrak{so}^*(2p)) \oplus \mathbb{R}iC \oplus \mathbb{R}iD$

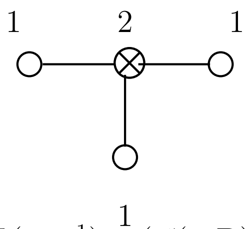


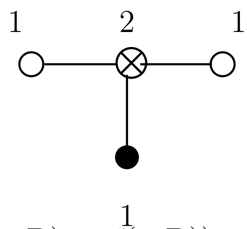


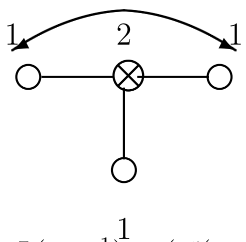
$L(t, t^{-1}) \otimes (\mathfrak{sp}(m, \mathbb{R}) \oplus \mathfrak{so}(p, q)) \oplus \mathbb{R}iC \oplus \mathbb{R}iD$

(v) $D(2, 1; \alpha)^{(1)}$

$L(t, t^{-1}) \otimes (\mathfrak{su}(2) \oplus su(2) \oplus \mathfrak{sl}(2, \mathbb{R}))$

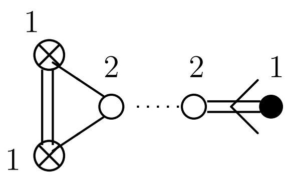




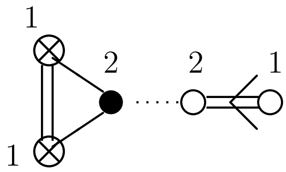


$L(t, t^{-1}) \otimes (\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}))$ $L(t, t^{-1}) \otimes (\mathfrak{sl}(2, \mathbb{C} \oplus \mathfrak{sl}(2, \mathbb{R}))$

(vi) $C(n)^{(1)}$

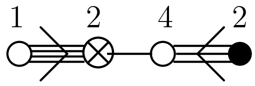


$L(t, t^{-1}) \otimes (\mathfrak{sp}(n, \mathbb{R}) \oplus \mathfrak{so}(2)) \oplus \mathbb{R}ic \oplus \mathbb{R}id$

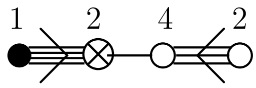


$L(t,t^{-1}) \otimes (\mathfrak{sp}(r,s) \oplus \mathfrak{so}(2)) \oplus \mathbb{R}ic \oplus \mathbb{R}id$

(vii) $G(3)^{(1)}$

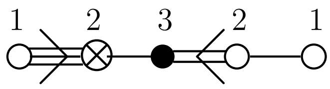


$L(t,t^{-1}) \otimes (\mathfrak{sl}(2,\mathbb{R}) \oplus g_c) \oplus \mathbb{R}ic \oplus \mathbb{R}id$

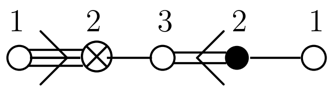


$L(t,t^{-1}) \otimes (\mathfrak{sl}(2,\mathbb{R}) \oplus g_s) \oplus \mathbb{R}ic \oplus \mathbb{R}id$

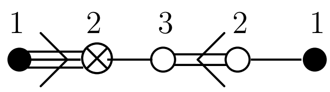
(viii) $F(4)^{(1)}$



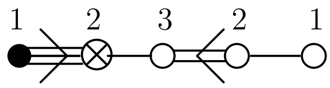
$L(t,t^{-1}) \otimes (\mathfrak{su}(2,\mathbb{R}) \oplus \mathfrak{so}(1,6)) \oplus \mathbb{R}iK \oplus \mathbb{R}iD$



$L(t,t^{-1}) \otimes (\mathfrak{su}(2,\mathbb{R}) \oplus \mathfrak{so}(2,5)) \oplus \mathbb{R}iK \oplus \mathbb{R}iD$



$L(t,t^{-1}) \otimes (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3,4)) \oplus \mathbb{R}iK \oplus \mathbb{R}iD$



$$L(t,t^{-1})\otimes(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{so}(7))\oplus\mathbb{R}iK\oplus\mathbb{R}iD$$

Chapter 5

Vogan diagram of twisted affine Lie superalgebra

A Vogan diagram is a Dynkin diagram with a Cartan involution of twisted affine superlagebras based on maximally compact Cartan subalgebras. In this chapter we construct the Vogan diagrams of twisted affine superalgebras. It is a part of completion of classification of vogan diagrams to superalgebras cases.

5.1 Realization of twisted Affine Lie superalgebras

Let $\mathcal{G}$ be a basic simple Lie superalgebra with non degenerate invariant bilinear form $(.,.)$ and σ be an automorphism of finite order $m > 1$. The eigenvalues of σ are of the form $e^{\frac{2\pi ki}{m}}$, $k \in \mathbb{Z}_m$ and hence admits the following $\mathbb{Z}_m$ grading:

$$\mathcal{G} = \bigoplus_{k=0}^{m-1} \mathcal{G}_k, m \geq 2$$

such that

$$[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j},\ i+j = i+j(\mod m)$$

and

$$\mathcal{G}_k = (\mathcal{G}_k)_{\bar{0}} \oplus (\mathcal{G}_k)_{\bar{1}}$$

$$\mathcal{G}_k = \{x \in \mathcal{G} | \sigma(x) = e^{\frac{2\pi k i}{m} . x}\}$$

The twisted affine Lie superalgebra is defined to be

$$\mathcal{G}^{(m)} = \left(\bigoplus_{k \in \mathbb{Z}_m} \mathbb{C}t^k \otimes \mathcal{G}_{k(\bmod m)} \right) \oplus \mathbb{C}K \oplus \mathbb{C}D$$

The Lie superalgebra structure on $\mathcal{G}^{(m)}$ is such that K is the canonical central element and

$$[x \otimes t^m + \lambda D, y \otimes t^n + \lambda_1 D] = ([x,y] \otimes t^{m+n} + \lambda n y \otimes t^n - \lambda_1 m x \otimes t^m + m\delta_{m,-n}(x,y)K$$

where $x, y \in \mathcal{G}^{(m)}$ and $\lambda, \lambda_1 \in \mathbb{C}$. The element D acts diagonally on $\mathcal{G}$ with interger eigenvalues and induces $\mathbb{Z}$ gradation.

5.2 Cartan Involution

Let $\mathfrak{g}$ be a compact Lie algebra if the group Int$\mathfrak{g}$ is compact. An involution θ of a real semisimple Lie algebra $\mathfrak{g}_0$ such that symmetric bilinear form

$$B_\theta(X, Y) = -B(X, \theta Y)$$

is positive definite is called a Cartan involution.

5.3 Cartan Involution of Contragradient Lie superalgebras

Let B be a supersymmeytic nondegenerate invariant bilinear form on $\mathcal{G}$. Define

$$B_\theta(X,Y) = B(X,\theta Y)$$

We say that a real form of $\mathcal{G}$ has Cartan automorphism $\theta \in \mathrm{aut}_{2,4}(\mathcal{G})$ if B restricts to the Killing form on $\mathcal{G}_0$ and B_θ is symmetric negative definite on $\mathcal{G}^{(m)}$.

The bilinear form $(.,.)$ on $\mathcal{G}$ gives rise to a nondegenerate symmetric invariant form on $\mathcal{G}^{(m)}$ by

$$B^{(m)}(\mathbb{C}[t,t^{-1}] \otimes \mathcal{G}, \mathbb{C}K \oplus \mathbb{C}d) = 0$$

. and this implies that

$$B^{(m)}(\bigoplus_{j\in\mathbb{Z}} t^j \otimes \mathcal{G}(\sigma)_{jmodm}, \mathbb{C}K \oplus \mathbb{C}D) = 0$$

$$B^{(m)}(t^j \otimes X, t^k \otimes Y) = \lambda\delta^{j+k,0}B(X,Y)$$

$$B^{(m)}(t^j \otimes X, K) = B^{(m)}(t^j \otimes X, D) = B^{(m)}(K,K) = B^{(m)}(D,D) = 0$$

$$B^m(K,D) = 1$$

Proposition 5.3.1. *Let $\theta \in aut_{2,4}(\mathcal{G}^{(m)})$. Then there exists a real form $\mathcal{G}_{\mathbb{R}}^{(m)}$ such that θ restricts to a Cartan automorphism on $\mathcal{G}_{\mathbb{R}}^{(m)}$.*

Proof. The proof of the proposition is similar to untwisted case. □

5.4 Root systems

We have mentioned the lowest root because it has the relationship with Kac-Dynkin label. We can get canonical nontrivial Kac-Dynkin labels by lowest root from the fundamental representation.

The simple root systems of twisted affine Lie superalgebra $\mathfrak{osp}(2m|2n)^{(2)}$ is given by

$$\Pi = \{\frac{k}{2} - \delta_1, \delta_1 - \delta_2, \cdots, \delta_{n-1} - \delta_n, \delta_n - e_1, e_1 - e_2, \cdots, e_{m-1} - e_m, e_m\}$$

. The $\mathcal{G}_0$ representation $\mathcal{G}_1$ is the fundamental representation of $\mathfrak{osp}(2m-1|2n)$ whose lowest weight is $-\delta_1$. For root systems of twisted affine Lie superalgebra $\mathfrak{osp}(2|2n)^{(2)}$, there exist an automorphism τ such that the invariant subsuperalgebra $\mathcal{G}_0$ is $\mathfrak{osp}(1|2n)$. The simple root system of $\mathcal{G}_0$ is

$$\Pi = \{\delta_1 - \delta_2, \cdots, \delta_{n-1} - \delta_n, \delta_n\}$$

The lowest weight of the $\mathcal{G}_1$ representation of $\mathcal{G}_0$ is δ_1. Similarly for twisted affine Lie superalgebra $\mathfrak{sl}(1|2n+1)^{(4)}$, we know the invariant subalgebra can be taken to as $\mathfrak{o}(2n+1)$ and the lowest weight is $-\delta_1$.

5.5 Vogan diagrams of affine Lie superalgebras

Let c be the circling of vertices (we are using circling instead of painting due to already existence of painted vertices in the Dynkin diagram), d be a diagram involution, a_s be a numerical labeling and D be a Dynkin diagram of $\mathcal{G}^{(m)}$. S is defined to be the set of d orbit vertices. [9]

Definition 5.5.1. A Vogan diagram (c, d) on D and one of the following holds:

(i) d fixes grey vertices

(ii) $\sum_S a_\alpha$ is odd.

The γ, δ and c are expressed in terms of the bases given as follows

$$\gamma = \sum_{i=1}^{n} a_i \alpha_i, \qquad \delta = \sum_{i=0}^{n} a_i \alpha_i$$

Fix a set π of simple roots of $\mathcal{G}$, we take $\hat{\pi} = \{\alpha_0 = \delta - \gamma\} \cup \pi$ to be the simple roots of $\mathcal{G}^{(m)}$ (γ is the highest weight in $\triangle_{\bar{0}}^{(1)} \cup \triangle_{\bar{1}}^{(1)}$).

If θ extends to $aut_{2,4}$ (automorphism of order 2 or 4) then θ permutes the extreme weight spaces $\mathcal{G}^{(m)}$. Since $\theta|_{\mathcal{G}_0}$ is represented by (c, d) on D_0 (even part (set of even roots) of the Dynkin diagram), it permutes the simple root spaces of $\mathcal{G}_0$. Hence θ permutes the lowest weight spaces of $\mathcal{G}^{(m)}$ and d extends to $inv(\mathcal{G}^{(m)})$ (where inv is involution on (.)).

Proposition 5.5.2. *Let $\mathcal{G}_{\mathbb{R}}$ be a real form, with Cartan involution $\theta \in inv(\mathcal{G}_{\mathbb{R}})$ and Vogan diagram (c, d) of D_0. The following are equivalent:*

(i) θ extends to $aut_{2,4}(\mathcal{G}^{(m)})$.

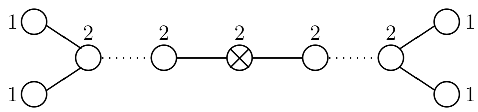
(ii) $(\mathcal{G}_{\bar{0}\mathbb{R}})$ extends to a real form of $\mathcal{G}^{(m)}$.

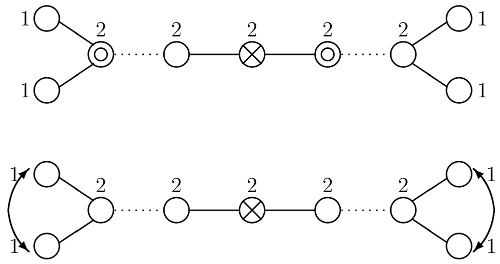
(iii) (c, d) extends to a Vogan diagram on D

Proof. The proof of this proposition follows from Proposition □

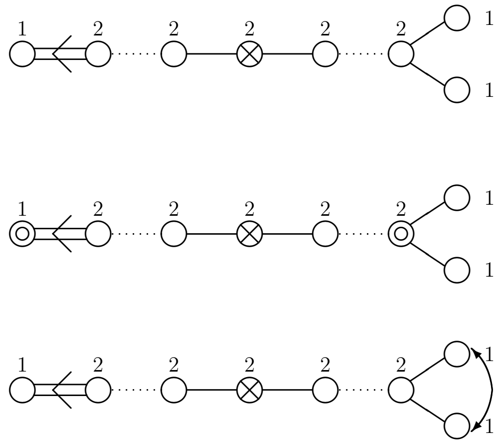
When there is a σ stable compact Cartan subalgebra then the Vogan diagrams are the following.

The Vogan diagrams of $\mathfrak{sl}(2m|2n)^{(2)}$ are

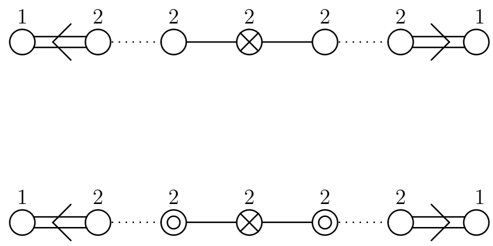




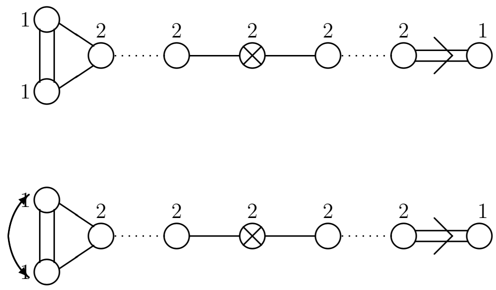
The Vogan diagrams of $\mathfrak{sl}(2m+1|2n)^{(2)}$ are

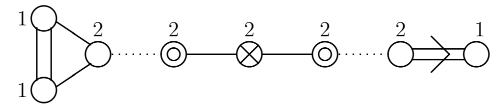


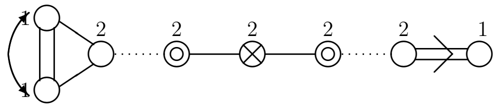
The Vogan diagrams of $\mathfrak{sl}(2m+1|2n+1)^{(2)}$ are



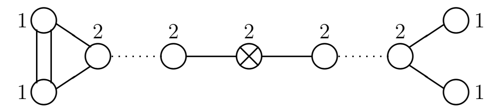
The Vogan diagrams of $\mathfrak{sl}(2|2n+1)^{(2)}$ are

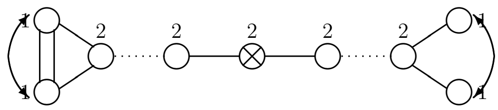




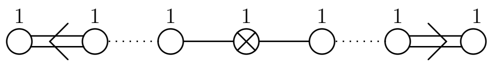


The Vogan diagrams of $\mathfrak{sl}(2|2n)^{(2)}$ are

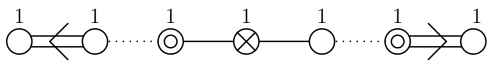




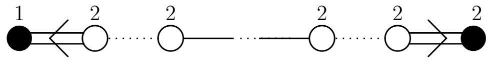
The Vogan diagrams of $\mathfrak{osp}(2m|2n)^{(2)}$ are



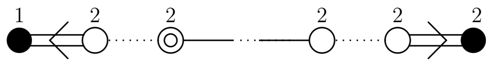
$\vdots$



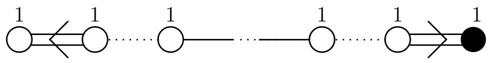
The lowest weight representation $\mathcal{G}_1$ of $\mathcal{G}_0$ is $-\delta_1$ and that makes the following Dynkin diagram for $\mathfrak{osp}(2|2n)^{(2)}$. The Vogan diagrams of $\mathfrak{osp}(2|2n)^{(2)}$ are



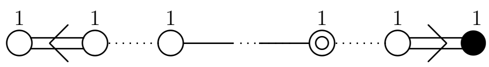
$\vdots$



The lowest weight representation $\mathcal{G}_1$ of $\mathcal{G}_0$ is $-\delta_1$ and that makes the following Dynkin diagram for $\mathfrak{sl}(1|2n+1)^{(4)}$. The Vogan diagrams of $\mathfrak{sl}(1|2n+1)^4$ are



$$\vdots$$



Chapter 6

Splints

In our last Chapter we discuss a little about the splints and flip Dynkin diagrams which are associated with the root systems of corresponding Lie superalgebras. In this Chapter we classify the splints of the root system of classical Lie superalgebras as a superalgebraic conversion of the splints of classical root systems. It can be used to derive branching rules, which have potential physical application in theoretical physics.

6.1 Characters and supercharacters

Consider $\mathcal{V}(\Lambda)$ a highest weight representation of $\mathfrak{h}$ with highest weight Λ, then

$$\mathcal{V}(\Lambda) = \underset{\lambda}{\oplus} \mathcal{V}_\lambda$$

where $\mathcal{V}_\lambda = \{\overrightarrow{v} \in \mathcal{V} | h(\overrightarrow{v}) = \lambda(h)\overrightarrow{v}, h \in \mathfrak{h}\}$

Let e^λ be the formal exponential, function on $\mathfrak{h}^*$ (dual of $\mathfrak{h}$) such that $e^\lambda(\mu) = \delta_{\lambda,\mu}$ where δ is Kronecker delta symbol, for two elements $\lambda, \mu \in \mathfrak{h}^*$, which satisfies

$e^{\lambda}e^{\mu} = e^{\lambda+\mu}$. Then the character and supercharacter of $\mathcal{V}(\Lambda)$ are defined by

$$\text{ch } \mathcal{V}(\Lambda) = \sum_{\lambda} (\dim \mathcal{V}_{\lambda}) e^{\lambda}$$

$$\text{sch } \mathcal{V}(\Lambda) = \sum_{\lambda} (-1)^{\deg \lambda} (\dim \mathcal{V}_{\lambda}) e^{\lambda}$$

We will reproduce the theorem of denominator identity [34] which is our basic motivations for in solving the splints of Lie superalgebras.

Theorem 6.1.1. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a basic classical type Lie superalgebra having defect $d = min\{m, n\}$, where $\mathfrak{g} = A(d-1, d-1)$ is replaced by $gl(d, d)$. Let $\triangle$ be any set of positive roots. For any simple root system $S \in \mathcal{S}(\triangle)$, where $\mathcal{S}(\triangle)$ is the collection of maximal isotropic subsets of $\triangle^+$ and we have*

$$C \cdot e^{\rho} R = \sum_{w \in W_{\mathfrak{g}}} sgn(w) w \left(\frac{e^{\rho}}{\prod_{\gamma \in S} (1 + sgn(\gamma) e^{-[\![\gamma]\!]})} \right),$$

$$C \cdot e^{\rho} \check{R} = \sum_{w \in W_{\mathfrak{g}}} sgn'(w) w \left(\frac{e^{\rho}}{\prod_{\gamma \in S} (1 - e^{-[\![\gamma]\!]})} \right),$$

where

$$C = \frac{C_{\mathfrak{g}}}{\prod\limits_{\gamma \in S} \frac{ht(\gamma)+1}{2}},$$

and $C_{\mathfrak{g}} = |W_{\mathfrak{g}}/W^{\sharp}|$.

if $\text{def}(\mathfrak{g}) = 1$, and $S \in \mathcal{S}(\triangle)$, then S consists of a single simple root. Hence in this case $C = C_{\mathfrak{g}}$ holds. Therefore we have to deal only with Lie superalgebras of type $\mathfrak{sl}(m, n)$, $B(m, n)$, $D(m, n)$ with defect d. The explicit values of $C_{\mathfrak{g}}$ are the following.

$$C_{\mathfrak{g}} = \begin{cases} d! & \text{if } \mathfrak{g} = A(m-1,n-1), \\ 2^d d! & \text{if } \mathfrak{g} = B(m,n), \\ 2^d d! & \text{if } \mathfrak{g} = D(m,n),\ m > n, \\ 2^{d-1} d! & \text{if } \mathfrak{g} = D(m,n),\ n \geq m, \\ 1 & \text{if } \mathfrak{g} = C(n), \\ 2 & \text{if } \mathfrak{g} = D(2,1,a),\ F(4),\ G(3). \end{cases}$$

The above defect for different superalgebras quite essential for construction of splints.

The present section gives the motivation to construct the splints from the character formula which is a ingredient combination of even and odd part. Let L^{μ} be the integrable module of g with the highest weight μ. Let L_a^{ν} be the integrable a module with the highest weight ν. Then

$$L^{\mu}_{g\downarrow a} = (-1)^{\deg\ \nu} \bigoplus_{\nu \in P_a^+} b_{\nu}^{(\mu)} L_a^{\nu}$$

where P_a^+ is the dominant weight lattice.

$$ch(L^{\mu}) = \frac{\psi^{(\mu)}}{\psi^{(0)}} = \frac{\psi^{(\mu)}}{R} = \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho) - \rho}}{\prod_{\alpha \in \triangle^+} (1 - e^{-\alpha})}$$

$$ch(L_a^{\nu}) = \frac{\psi_a^{(\nu)}}{\psi_a^{(0)}} = \frac{\psi^{(\mu)}}{R_a} = \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho_a) - \rho_a}}{\prod_{\alpha \in \triangle_a^+} (1 - e^{-\alpha})}$$

$$ch(L^{(\mu)}) = \sum_{\nu \in P_a^+} b_{\nu}^{(\mu)} ch(L_a^{\nu})$$

$$\prod_{\alpha\in\triangle_1^+}(1-e^{-\alpha})^{\text{mult}(\alpha)}\prod_{\beta\in\triangle_2^+}(1-e^{\phi\circ\beta})^{\text{mult}(\beta)}=\prod_{\gamma\in\triangle_1^+}(1-e^{-\gamma})^{\text{mult}(\gamma)}$$

$$\triangle=\triangle_1\cup\triangle_2$$

$$A_\rho=\frac{\prod}{\alpha\in\triangle^+}(e^{\alpha/2}-e^{-\alpha/2})$$

$$A_\rho=A_1A_2$$

The above character formula shows, how the character formula decomposes similar to splints of the root systems.

6.2 Splints of Simple root system of Classical Lie superalgebras

Definition 6.2.1. Suppose $\triangle_0$ and $\triangle$ are root systems. Then embedding ι of a root system $\triangle$ is a bijective map of roots of $\triangle_0$ to a (proper) subset of $\triangle$, that commutes with vector composition law in $\triangle_0$ and $\triangle$. $\iota(\gamma)=\iota(\alpha)+\iota(\beta)$ for all α,β,$\gamma\in\triangle_0$ such that $\gamma=\alpha+\beta$.

$\iota(\gamma)\neq\iota(\alpha)+\iota(\beta)$ if $\alpha\in\triangle_1$ and $\beta\in\triangle_0$

Definition 6.2.2. A root system of Lie superalgebra Δ "splinters" as (Δ_1,Δ_2) if there are two embeddings $\iota_1:\Delta_1\hookrightarrow\Delta$ and $\iota_2:\Delta_2\hookrightarrow\Delta$ where

1. Δ is the disjoint union of the images of ι_1 and ι_2 and
2. neither the rank of Δ_1 nor the rank of Δ_2 equal to or greater than the rank of Δ.

It is equivalent to say that (Δ_1,Δ_2) is a splint of Δ. Each component Δ_1 and Δ_2 is a stem of the splint (Δ_1,Δ_2).

Proposition 6.2.3. *The splints of the root systems of Lie superalgebras is the splintering of root system of even parts.*

Proof. The splints of the case-1 to case-5 Lie superalgebras proved the proposition. We have restricted ourselves only to classical Lie superalgebras. $\square$

Case-1 : $A(m, n)$

Simple root system $\{\triangle_0 = \{e_i - e_j; 1 \leq i \neq j \leq m+1, \delta_i - \delta_j; 1 \leq i \neq j \leq n+1,$
$\triangle_1 = \{e_i - \delta_j; 1 \leq i \leq m+1; 1 \leq j \leq n+1\}$
Define

$$\triangle(A_m) = \{e_i - e_j)1 \leq i \neq j \leq m+1$$

$$\triangle(A_n) = \{\delta_i - \delta_j)1 \leq i \neq j \leq n+1$$

and

$$\triangle^{special} = \{e_i - \delta_j\}1 \leq i \leq m+1; 1 \leq j \leq n+1$$

Using the splints of A_m and A_n from [15] we get the following splints.

(i) The first splint of $A(m, n)$; $((mA_1, A_{n-1}) + (nA_1, A_{n-1}), \triangle^{special})$

$$(rA(1,1), A(n-1, n-1))$$

(ii) The second splint is $(A_1 + A_{m-1}, (m-1)A_1) + (A_1 + A_{n-1}, (n-1)A_1), \triangle^{special})$

$$(A(1,1) + A(m-1, n-1), (n-1)A(1,1))$$

Using the defect d we get the above splints.

(iii) The third splint is $((A(k, i), A(l, j))$ where $k + l = m$, $i + j = n$. The simple component of this splint is a maximal regular subsuperalgebra.

Proposition 6.2.4. *The splints of root systems of $A(m,n)$ is equivalent to splints of the even root systems.*

Case 2 : $B(m,n)$, $m > 0$ or $osp(2m+1, 2n)$

Simple root system

$$\{\delta_1 - \delta_2, \delta_2 - \delta_3, \cdots, \delta_n - e_1, e_1 - e_2, \cdots, e_{m-1} - e_m, e_m\}$$

$$\triangle(B_m) = \{e_i \pm e_j, e_i\} = (D_m, mA_1)\triangle(B_4) = \{e_i \pm e_j, e_i\} = (D_4, 4A_1)$$

The only splints of D_m is for the value of $m = 4$; the available splint is $(2A_2, 2A_2)$.

$$\triangle(C_n) = \{\delta_k \pm \delta_l, 2\delta_k\} = (nA_1, D_n), \triangle^{special} = \{\delta_j - e_i\}$$

The splint of the first superalgebra $B(m,n)$ is $(4A(2,2), 4A(1,1))$

(i) The first splint of $B(m,n)$ is $((B_m, mA_1) + (nA_1, D_n), \triangle^{special})$

(ii) The second splint for higher values of m and n is $(B(k,i) + D(i,j), \triangle^{special})$

Example 6.2.5. Splints of $B(4,4)$

$$\triangle = \{e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3, \delta_1 \pm \delta_2, 2\delta_1, 2\delta_2, e_1, e_2, e_3\}$$

$$\triangle_1 = \{e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3, e_1, e_2, e_3\}$$

$$\triangle_2 = \{\delta_1 \pm \delta_2, 2\delta_1, 2\delta_2\}$$

We get the splint $(4A(2,2), 4A(1,1))$

Case 3 : $B(0,n)$

Simple root system: $\{\delta_1-\delta_2,\delta_2-\delta_3,\cdots,\delta_{m-1}-\delta_m,\delta\}$

$$\Delta_1=\{\pm\delta_i\pm\delta_j;\pm2\delta_i\}\,,$$

$$\Delta_2=\{\pm\delta_i\}$$

Case 4 : $C(n+1)$

Simple root system:

$$\{e-\delta_1,\delta_1-\delta_2,\cdots,\delta_{n-1}-\delta_n,2\delta_n\}$$

so

$$\Delta_1=\{\delta_1-\delta_2,\delta_2-\delta_3,\cdots,\delta_{m-1}-\delta_m,\delta\}$$

and

$$\Delta_2=\{e-\delta\}$$

$$\Delta_1=\{\pm\delta_1\pm\delta_j;\pm2\delta_i\}\,,$$

$$\Delta_2=\{\pm e\pm\delta_i\}$$

The possible splint will be $(B(0,n),C(1))$

Case 5 : $D(m,n)$

Simple root system: $\{\delta_1-\delta_2, \delta_2-\delta_3, \cdots, \delta_{n-1}-\delta_n, \delta_n-e_1, e_1-e_2, \cdots, e_{m-2}-e_{m-1}, e_{m-1}-e_m, e_{m-1}+e_m\}$

$$\Delta_1 = \{\pm e_i \pm e_j; \pm\delta_i \pm \delta_j; \pm 2\delta_i\}$$

$$(i \neq j)\,, \Delta_2 = \{\pm\delta_i\}$$

$$\triangle(D_m) = \{e_i \pm e_j\}$$

$$\triangle(C_n) = \{\delta_k \pm \delta_j, 2\delta_i\}$$

There is only one value of $m \geq 4$ for which D_m splints is 4. We get the splint $((4A(2,2), 4A(1,1)))$

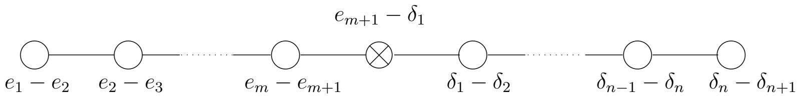
6.3 Defining relations and Flip Dynkin Super diagrams

Definition 6.3.1. Fliping the fermions and bosonic root gives a filp Dynkin superdiagram. Flip is a map f

$$f: \begin{array}{l} \bigcirc \to \otimes \\ \otimes \to \bigcirc \end{array}$$

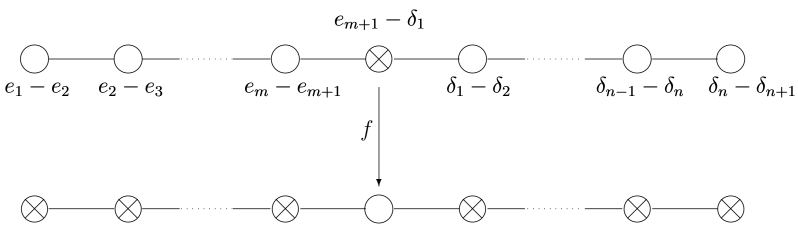
If there is no more than one line between the roots of the Dynkin diagrams (simply laced Dynkin diagram). Thus fliping preserve the line and orientation of doubly and triply connected Dynkin diagram. Flip Dynkin Superdiagram creats flip Cartan matrix.

Lie superalgebra $\mathfrak{sl}(m,n)$:

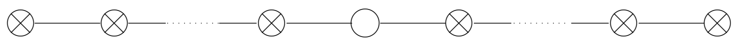


the corresponding Cartan matrix is

$$\begin{pmatrix} 2 & -1 & & & & & \\ -1 & \ddots & \ddots & & & & \\ & \ddots & 2 & -1 & & & \\ & & -1 & 0 & 1 & & \\ & & & 1 & -2 & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & -2 \end{pmatrix}$$



So the supersymmetry partner of $\mathfrak{sl}(m,n)$ is the flip Dynkin Superdiagram

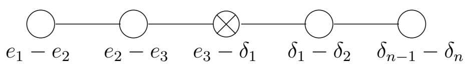


corresponds Cartan matrix with the normalizations $(e_i, e_j) = \delta_{ij}(i, j = 1, \cdots, m+1)$, $(\delta_k, \delta_l) = -\delta_{kl}(k, l = 1, \cdots, n+1)$, $(e_i, \delta_k) = 0$ and the flip Cartan matrix become

$$\begin{pmatrix} 2 & -1 & & & & & \\ -1 & \ddots & \ddots & & & & \\ & \ddots & 2 & -1 & & & \\ & & -1 & 0 & 1 & & \\ & & & 1 & -2 & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & & & & & 1 & -2 \end{pmatrix} \xrightarrow{f} \begin{pmatrix} 0 & -1 & & & & & \\ -1 & \ddots & \ddots & & & & \\ & \ddots & 0 & -1 & & & \\ & & -1 & -2 & 1 & & \\ & & & 1 & 0 & \ddots & \\ & & & & \ddots & \ddots & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$

Example 6.3.2. $\mathfrak{sl}(2,2)$

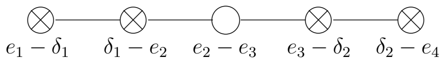
The Dynkin diagram is



Corresponding Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

The associated flip dynkin superdiagram



Corresponding Cartan Matrix is

$$CM1 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \xrightarrow{f} = CM2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

6.4 Serre-type relations

$[H_i, H_i] = 0$, $[X_i^+, X_j^-] = \delta_{ij} H_i$,$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}$ for $H_i = [X_i^+, X_i^-]$

Proposition 6.4.1 ([7]). *The number k_{in} and k_{ni} are expressed in terms of B_{kj} as follows*

$$(ad_{X_k^{\pm}})^{1+B_{kj}}(X_j^{\pm}) = 0 \quad for\ k \neq j$$

$$[X_i^{\pm}, X_i^{\pm}] = 0 \quad if\ A_{ii} = 0$$

The above relations will be called Serre relations of the Lie superalgebras.

Here

$$B_{kj} = \begin{cases} -\frac{2A_{kj}}{A_{kk}} & \textit{if } A_{kk} \neq 0 \textit{ and } -\frac{2A_{kj}}{A_{kk}} \in \mathbb{Z}_+, \\ 1 & \textit{if } i_k = \bar{1}, A_{kk} = 0, A_{kj} \neq 0 \\ 0 & \textit{if } i_k = \bar{1}, A_{kk} = A_{kj} = 0 \\ 0 & \textit{if } i_k = \bar{0}, A_{kk} = 0, A_{kj} = 0 \end{cases}$$

Defining relations for Cartan matrix CM1 are

$[x_1, x_3] = 0$

$[x_1, x_4] = 0$

$[x_1, x_5] = 0$

$[x_2, x_4] = 0$

$[x_3, x_3] = 0$

$[x_3, x_5] = 0$

$[x_1, [x_1, x_2]] = 0$

$[x_2, [x_1, x_2]] = 0$

$[x_2, [x_2, x_3]] = 0$

$[x_4, [x_4, x_5]] = 0$

$[[x_2, x_3], [x_3, x_4]] = 0$

$[\{x_1, x_2, x_3, x_4, x_5, [x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5], [x_3[x_1, x_2]]$

$[x_4[x_2, x_3]]$,$[x_4[x_3, x_4]]$,$[x_5[x_3, x_4]]$,$[x_4[x_4, [x_2, x_3]]]$,$[x_4[x_4, [x_3, x_4]]]$, $[[x_1, x_2], [x_3, x_4]]$,

$[[x_2, x_3], [x_3, x_4]]$,$[[x_2, x_3], [x_4, x_5]]$,$[[x_3, x_4], [x_3, x_4]]$,$[[x_3, x_4], [x_4, x_5]]$,

$[x_4, [x_4, [x_4, [x_2, x_3]]]]$,$[x_4, [x_4, [x_4, [x_3, x_4]]]]$,$[x_1, x_2], [x_4, [x_4, [x_3, x_4]]]]$,

$[x_3, x_4], [x_4, [x_4, [x_2, x_3]]]]$,$[x_3, x_4], [x_4, [x_4, [x_3, x_4]]]]$,

$[[x_4, x_5], [x_4, [x_4, [x_2, x_3]]]]$,$[[x_4, x_5], [x_4, [x_4, [x_2, x_3]]]]$,$[[x_4, x_5], [x_4, [x_4, [x_3, x_4]]]]$,

$[[x_4, x_5], [x_1, x_2], [x_3, x_4]]]]$,

$[[x_4, x_5], [x_3, x_4], [x_4, x_5]]]]$,$[[x_4, x_5], [x_3, x_4], [x_4, x_5]]]]$,$[[x_4, x_5], [x_3, x_4], [x_3, x_4]]]]$,

$[[x_4, x_5], [x_3, x_4], [x_3, x_4]]]]$,

$[[x_4, x_5], [x_3, x_4], [x_4, x_5]]]],$

$[[x_3, [x_1, x_2]], [x_4, [x_3, x_4]]],[[x_3, [x_1, x_2]], [x_4, [x_3, x_4]]],[[x_4, [x_2, x_3]],$

$[x_4, [x_2, x_3]]],[[x_4, [x_2, x_3]], [x_4, [x_3, x_4]]],$

$[[x_4, [x_2, x_3]], [x_5, [x_3, x_4]]],[[x_4, [x_3, x_4]], [x_4, [x_3, x_4]]],[[x_4, [x_3, x_4]],$

$[x_5, [x_3, x_4]]],[[x_5, [x_3, x_4]], [x_5, [x_3, x_4]]]$

Defining relations of Cartan Matrix CM2

$[x_1, x_1] = 0$

$[x_1, x_3] = 0$

$[x_1, x_4] = 0$

$[x_1, x_5] = 0$

$[x_2, x_2] = 0$

$[x_2, x_4] = 0$

$[x_2, x_5] = 0$

$[x_3, x_5] = 0$

$[x_4, x_4] = 0$

$[x_5, x_5] = 0$

$[x_3, [x_2, x_3]] = 0$

$[x_3, [x_3, x_4]] = 0$

$[x_3, [x_2, x_3]] = 0$

$[[x_3, x_4], [x_4, x_5]] = 0$

$[\{x_1, x_2, x_3, x_4, x_5, [x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5], [x_3, [x_1, x_2]],$

$[x_4, [x_2, x_3]],[x_5, [x_3, x_4]],[[x_1, x_2], [x_2, x_3]],[[x_1, x_2], [x_3, x_4]],$

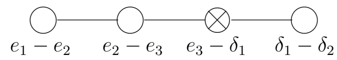
$[[x_2, x_3], [x_4, x_5]],[[x_1, x_2], [x_3, [x_1, x_2]]],[[x_1, x_2], [x_4, [x_2, x_3]]],$

$[[x_4, x_5], [x_3, [x_1, x_2]]],[[x_1, x_2], [[x_1, x_2], [x_2, x_3]]],[[x_1, x_2],$

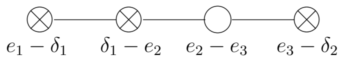
$[[x_1, x_2], [x_3, x_4]]],[[x_4, x_5], [[x_1, x_2], [x_2, x_3]]],$

$[[x_3, [x_1, x_2]], [x_4, [x_2, x_3]]]\}]$

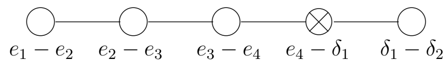
For $\mathfrak{sl}(2,1)$ Dynkin diagram is



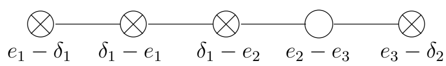
So the filp Dynkin superdiagram becomes



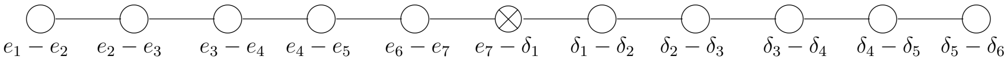
For $\mathfrak{sl}(3,2)$ Dynkin diagram is



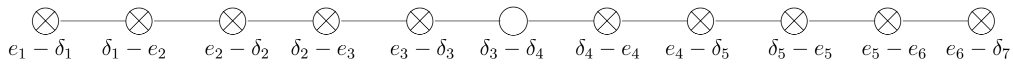
the filp Dynkin superdiagram becomes



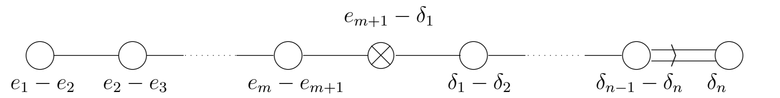
For $\mathfrak{sl}(5,5)$ Dynkin diagram is



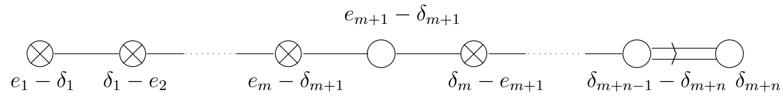
corresponding flip Dynkin diagram is



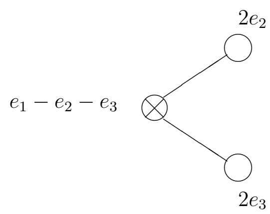
$\mathfrak{osp}(2m+1, 2n)$:



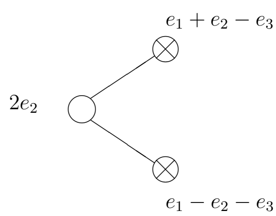
The flip Dynkin Superdiagram is



case $D(2,1;\alpha)$



With the normalizations $(e_1, e_1) = \frac{-(1+\alpha)}{2}$, $(e_2, e_2) = \frac{1}{2}$, $(e_3, e_3) = \frac{\alpha}{2}$, flip Dynkin Superdiagram becomes



since $(e_1+e_2-e_3, e_1+e-e_3) = 0$ and $(e_1-e_2-e_3, e_1-e-e_3) = 0$.

case $G(3)$

There is a trivial flip in the dynkin diagram of $G(3)$ to preserve the orientation and the connected lines between roots. Thus

$$\otimes - \bigcirc \Lleftarrow \bigcirc \xrightarrow{f} \otimes - \bigcirc \Lleftarrow \bigcirc$$

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Biswajit Ransingh

Ph.D. Student in Mathematics
Roll No. 507MA001
National Institute of Technology
Rourkela - 769008 (India)